ADJOINT LINEAR DIFFERENTIAL OPERATORS

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In this note we derive a new vector-matrix formulation of ordinary linear differential equations. We then use this formulation to motivate a generalization of the classical ordinary linear differential operator and its adjoint. Among the advantages resulting from the use of these generalized operators are: The operator and its adjoint "have the same form," no differentiability conditions are required on the coefficients, the Lagrange identity and the adjointness condition for boundary value problems are very simple.

Denote by C the set of continuous complex-valued functions on an interval I. Let $p_i \in C$, $i=0, \cdots, k-1$. Consider $Lx = \sum_{i=0}^{k} p_i x^i$ where $p_k = 1$.

THEOREM 1. *There is a* $k \times k$ *matrix* $F=(f_{ij})$ *with the properties*:

(i) $f_{ij} \in C$,

(ii) $f_{i,i+1}(t)=1$, $i=1, \cdots, k-2$, $f_{k-1,k}(t) \neq 0$, $t \in I$,

and

(iii) $f_{ij}=0$ *if*

a. $j>i+1$ *or*

b. $i<k-1$ *and* $j \leqq i$ *or*

c. $i=k-1$ *or* $i=k$ *and* $i+j$ *is even*;

such that $Lx=0$ *is equivalent to the vector-matrix equation* $X'=F \cdot X$ *in the following sense*:

Given a solution x *of* $Lx=0$ *the vector* $X=(x_i)$ *where* $x_1=x$, $x_i=x_{i-1}'$, $i=2, \cdots, k-1$ *and* $x_k = \exp(\int p_{k-1})x^{k-1} + [\int \exp(\int p_{k-1})p_{k-3}]x^{k-3} + [\int \exp(\int p_{k-1})p_{k-5}]x^{k-5} + \cdots$ *is a solution of* $X'=F \cdot X$ *and conversely, given a solution* $X=(x_i)$ *of* $X'=F \cdot X$ *its first component* x_1 *is a solution of* $Lx=0$.

PROOF. Let $r_i=\exp(\int p_{k-1}) \cdot p_i$, $i=0, 1, \cdots, k-2$, $r_{k-1}=\exp(\int p_{k-1})$. Define $F=(f_{ij})$ by:

Case 1. k even.

a. $f_{i+1}(t)=1$, $i=1, \cdots, k-2$, $f_{k-1,k}=r_{k-1}^{-1}$,

b. $f_{k-1,p}=-r_{k-1}^{-1}\int r_{p-1}$, $p=2, 4, \cdots, k-2$,

c. $f_{k1}=-r_0$, $f_{kp}=-[r_{p-1}-\int r_{p-2}]$, $p=3, 5, \cdots, k-1$

and all other $f_{ij}=0$.

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Case 2. k odd.

a. $f_{i,i+1}=1$, $i=1,\cdots,k-2$, $f_{k-1,k}=r_{k-1}^{-1}$,

b. $f_{k-1,p}=-r_{k-1}^{-1}\int r_{p-1}$, $p=1,3,\cdots,k-2$,

c. $f_{kp}=-[r_{p-1}-\int r_{p-2}]$, $p=2,4,\cdots,k-1$

and all other $f_{ij}=0$.

Straightforward manipulations show that F has the properties stated by the theorem.

Let $F=(f_{ij})$ denote a $k\times k$ matrix such that

(i) $f_{ij}\in C$,

(ii) $f_{ij}=0$ if $i+j$ is even or $j>i+1$,

(iii) $f_{i,i+1}(t)\neq 0$ for $t\in[a,b]$, $i=1,\cdots,k-1$.

Let

$$D_0u=u$$

$$D_iu=f_{i,i+1}^{-1}\left[(D_{i-1}u)'-\sum_{j=1}^{i}f_{ij}D_{j-1}u\right],\ i=1,\cdots,k-1,$$

for all u for which the right-hand side is defined on I.

Let $Lu=(D_{k-1}u)'-\sum_{i=1}^{k}f_{ki}D_{i-1}u$ for all $u\in U=\{u\ni(D_{k-1}u)'$ exists on $I\}$.

Let $D_0^+v=v$ and for $i=1,\cdots,k-1$,

$$D_i^+v=\bar{f}_{k-i,k+1-i}^{-1}\left[(D_{i-1}^+v)'-\sum_{j=1}^{i}\bar{f}_{k+1-j,k+1-i}D_{j-1}^+v\right]$$

for all v for which the right side is defined on I.

Let $L^+v=(D_{k-1}^+v)'-\sum_{j=1}^{k}\bar{f}_{k+1-j,1}D_{j-1}^+v$ for $v\in V=\{v\ni(D_{k+1}v)'$ exists on $I\}$.

Let $Tu=(D_iu)$, $i=0,\cdots,k-1$, $u\in U$, $T^+v=(D_i^+v)$, $i=0,\cdots,k-1$ $v\in V$, and $Z=((-1)^{k+i+j+1}\delta_{i,k+1-j})$ where δ is the Kronecker delta.

Routine calculations establish

THEOREM 2. $(Lu)v+(-1)^{k+1}uL^+v=(ZTu,\ T^+v)'$ *for each* $u\in U$, $v\in V$.

By an argument similar to the one given in [1, Theorem 3.1, Chapter 11] we obtain

THEOREM 3. *The boundary conditions* $ATx(a)+BTx(b)=0$ *and* $PT^+x(a)+QT^+x(b)=0$ *where* A *and* B *are* $m\times k$ *matrices,* $1\leqq m<2k$, *such that* $(A:B)$ *has rank* m *and* P, Q *are* $(2k-m)\times k$ *matrices such that* $(P:Q)$ *has rank* $2k-m$ *are adjoint (for definition see* [1, *p.* 289]) *if and only if* $AZ^{-1}P^*=BZ^{-1}Q^*$.

Reference

1. E. A. Coddington and N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill, New York, 1955.

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LOWER BOUNDS FOR SOLUTIONS OF DIFFERENTIAL INEQUALITIES IN HILBERT SPACE

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Let A be an operator in a Hilbert space and let $u(t)$ be in the domain of A for each $t\in[0,\infty)$. Assuming u is strongly differentiable, Au strongly continuous and du/dt strongly piecewise continuous, all with respect to t, we define

$$Lu = \frac{du}{dt} - Au. \tag{1}$$

In the case where A is symmetric, i.e., $(Au, v) = (u, Av)$, Cohen and Lees [1] obtained lower bounds for solutions of differential inequalities of the form

$$|Lu(t)| \leqq \phi(t)\,|u(t)|. \tag{2}$$

They proved that if $\phi\in L_p(0,\infty)$ for some p with $1\leqq p\leqq 2$, then any solution of (2) such that $u(0)\neq 0$ satisfies

$$|u(t)| \geqq Ke^{\lambda t},$$

where $K>0$ and λ are constants depending on the solution. Assuming that A is selfadjoint, Agmon and Nirenberg [2] found a simpler

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