# ITERATED $w^*$-SEQUENTIAL CLOSURE OF A BANACH SPACE IN ITS SECOND CONJUGATE

R. D. McWILLIAMS[1]

1. If $X$ is a real Banach space and $S$ is a subspace of the second conjugate space $X^{**}$, let $K(S)$ be the $w^*$-sequential closure of $S$ in $X^{**}$; thus if $F \in X^{**}$, then $F \in K(S)$ if and only if $F$ is the $w^*$-limit of a sequence in $S$. If $J$ is the canonical mapping from $X$ into $X^{**}$, then $K(JX)$ is norm-closed in $X^{**}$ [3]. In the present paper it is shown that $K(K(JX))$ can fail to be norm-closed in $X^{**}$, thus answering a question raised in [4]. The proof is a modification of the proof in [4] that $K(K(JP))$ can fail to be norm-closed in $X^{**}$ for $P$ a cone in $X$.

2. If $[a; b]$ is a finite real interval, let $B[a; b]$ be the Banach space of all bounded real functions $z$ on $[a;b]$ with $\|z\| = \sup\{|z(t)| : a \leqq t \leqq b\}$. If $A$ is a subset of $B[a; b]$, let $L(A)$ be the set of all $z \in B[a; b]$ such that $z$ is the pointwise limit of a bounded sequence in $A$. If $X$ is a closed subspace of the Banach space $\mathfrak{C}$ of all continuous functions on $[a; b]$, and if $\{x_n\}$ is a bounded sequence in $X$ which is pointwise convergent to a function $z \in L(X)$, Lebesgue's bounded convergence theorem [2, p. 110] shows that $\{Jx_n\}$ is $w^*$-convergent to an element $F \in K(JX)$ which depends only on $z$. The correspondence $z \rightarrow F$ thus obtained is an isometric isomorphism from $L(X)$ onto $K(JX)$, and for simplicity of notation we shall henceforth identify $L(X)$ and $K(JX)$. In like manner, $L(L(X))$ and $K(K(JX))$ are isometrically isomorphic and will be identified with each other.

The main result will be shown to follow from the following theorem.

THEOREM 1. *For each number $c \geqq 1$ there exists a Banach space $X$ with the property that there is an $x_0 \in K(K(JX))$ such that $\|x_0\| = 1$, but if $\{y^h\}$ is a sequence in $K(JX)$ which is $w^*$-convergent to $x_0$, then* $\liminf_h \|y^h\| \geqq c$.

PROOF. If $s$, $\alpha$, and $\beta$ are positive numbers such that $0 \leqq s - \alpha - \beta$ and $s + \alpha + \beta \leqq 3$, let $f(s; \alpha, \beta)$ be the continuous real function on $[0; 3]$ whose values at the points $0$, $s-\alpha-\beta$, $s-\alpha$, $s+\alpha$, $s+\alpha+\beta$, and $3$ are $0, 0, 1, 1, 0$, and $0$ respectively and which is linear on each of the five subintervals so determined.

---

Received by the editors December 18, 1964.

[1] Supported in part by National Science Foundation Grant GP-2179 and Florida State University Research Council Grant 036(42).

If $u, v, i, j$ are in the set $\omega$ of all positive integers, and $i<2^u$, let $s_{ui}=2^{-u}i$ and $t_{vj}=2-2^{-v}(1+2^{-j})$. By induction on $p$, it is possible to choose irrational positive numbers $c_{piq}$, where $p, i, q\in\omega$ and $i<2^p$, such that

(i) $c_{pi,q+1}=2^{-1}c_{piq}$;

(ii) $2\leqq 2+s_{pi}\pm 2c_{pi1}\leqq 3$;

(iii) if $i$ is odd, then $f(2+s_{p'j};\ c_{p'jq},\ c_{p'jq})$ is linear on the interval $[2+s_{pi}-2c_{pi1};\ 2+s_{pi}+2c_{pi1}]$ if $p'<p$ and $j<2^{p'}$;

(iv) $f(2+s_{pi};\ c_{pi1},\ c_{pi1})$ vanishes at the points $2+s_{p'j}\pm 2c_{p'j1}$ if $p'\leqq p$ and $j<2^{p'}$.

Let $X$ be the closed subspace of $\mathfrak{C}[0;3]$ generated by $\{x_{pq}:p,\ q\in\omega\}$, where

$$
\begin{aligned}
x_{pq} = {} & \sum_{i=1}^{2^p-1} f(s_{pi};\ 5\cdot 2^{-p-q-5},\ 2^{-p-q-5}) \\
& + c\sum_{v=1}^{p}\sum_{j=p}^{p+q-1} f(t_{vj};\ 2^{-v-j-q-2},\ 2^{-v-j-q-2}) \\
& + \sum_{i=1}^{2^p-1} f(2+s_{pi};\ c_{piq},\ c_{piq}).
\end{aligned}
\tag{2.1}
$$

For each subset $S$ of $[0;\ 3]$ let $\chi(S)$ be the characteristic function of $S$ relative to $[0;\ 3]$. For each fixed $p\in\omega$, the sequence $\{x_{pq}\}_{q\in\omega}$ is pointwise convergent to the function

$$
\begin{aligned}
x_p = {} & \chi(\{s_{pi}: i<2^p\}\cup\{2+s_{pi}: i<2^p\}) \\
& + c\chi(\{t_{vj}: v\leqq p\leqq j\}),
\end{aligned}
\tag{2.2}
$$

and the sequence $\{x_p\}_{p\in\omega}$ is pointwise convergent to the function

$$
x_0 = \chi(\{s_{pi}:p\in\omega,\ i<2^p\}\cup\{2+s_{pi}:p\in\omega,\ i<2^p\}). \tag{2.3}
$$

It is easily verified that the norm of each $x_{pq}$ and each $x_p$ is $c$, whereas $\|x_0\|=1$. Thus $\{x_p:p\in\omega\}\subseteq K(JX)$, and $x_0\in K(K(JX))$.

Now let $\{y^h\}_{h\in\omega}$ be an arbitrary sequence in $K(JX)$ which is $w^*$-convergent to $x_0$; we must show that $\liminf_h\|y^h\|\geqq c$. Each $y^h$ can be regarded as the pointwise limit of a bounded sequence $\{y^{hk}\}_{k\in\omega}$ in $X$. Without changing $\liminf_h\|y^h\|$, we can assume that $y^h(1/2)=1$ for each $h$. Since $\{x_{pq}:p,\ q\in\omega\}$ generates $X$, each $y^{hk}$ can be taken in the form

$$
y^{hk} = \sum_{p,q\in\omega} a_{pq}^{hk}x_{pq}, \tag{2.4}
$$

where for each pair $(h, k)$ only a finite number of the coefficients $a_{pq}^{hk}$ are different from zero. Moreover, by a proof in [3, pp. 333–334], for each $h$ the sequence $\{y^{hk}\}_{k\in\omega}$ can be chosen so that $\lim_k \|y^{hk}\| = \|y^h\|$.

Let $\epsilon > 0$ be given. For each $H \in \omega$ let

$$(2.5) \quad S_H = \bigcap_{h \geq H} \{t \in [0;1] : |y^h(t)| < \epsilon \text{ and } |y^h(2+t)| < \epsilon\}.$$

Since $\bigcup_{H\in\omega} S_H$ contains all but a countable number of the points of $[0; 1]$ and is hence of the second category, there exists an integer $H_0 \in \omega$ such that $S_{H_0}$ is dense in a closed interval $I$. There exist $p_0 \in \omega$ and an odd integer $i_0$, with $1 \leqq i_0 < 2^{p_0}$, such that the interval $I_0 = [2^{-p_0}i_0; 2^{-p_0}i_0 + 2^{-p_0-1}]$ is contained in $I$.

Now $x_0(t_{vv}) = 0$ for each $v \in \omega$. Hence there exists $H_1 \geqq H_0$ such that $|y^h(t_{vv})| < \epsilon c p_0^{-1}$ for all $h \geqq H_1$ and $v < p_0$.

Observe that $y^{hk}(t_{pp} - 2^{-2p-q-2}) - y^{hk}(t_{pp} - 2^{-2p-q-1}) = a_{pq}^{hk}$ in every case. Thus $a_{pq}^h \equiv \lim_k a_{pq}^{hk}$ exists for all $h$, $p$, $q$, and $\lim_h a_{pq}^h = 0$ for every pair $(p, q)$. Since $i_0$ is odd, there are only a finite number $N$ of pairs $(p, q)$ such that $p < p_0$ and $x_{pq}$ assumes nonzero values on $I_0$. Hence there exists $H_2 \geqq H_1$ such that $|a_{pq}^h| < \epsilon N^{-1}$ for each $h \geqq H_2$ and each pair $(p, q)$ such that $x_{pq}$ assumes nonzero values on $I_0$ and $p < p_0$.

Now fix an integer $h \geqq H_2$. Since $y^h$ is a Baire function of the first class, $y^h$ must have a point of continuity in the interval $G = \{2 + t : t \in I\}$; then since $|y^h(s)| < \epsilon$ for all $s$ in a dense subset of $G$, it follows that there is a closed subinterval $G_0$ of $G$ such that $|y^h(s)| < 2\epsilon$ for all $s \in G_0$. There exist $p_1 > p_0$ and an odd integer $i_1$, with $1 \leqq i_1 < 2^{p_1}$, such that $G_1 \equiv [2 + s_{p_1 i_1} - 2c_{p_1 i_1 1}; 2 + s_{p_1 i_1} + 2c_{p_1 i_1 1}] \subseteq G_0$. By the conditions on the numbers $c_{piq}$, each $x_{pq}$ with $p < p_1$ is linear on $G_1$, and each $x_{pq}$ with $p \geqq p_1$ vanishes at the endpoints of $G_1$. Hence for each $k \in \omega$,

$$(2.6) \quad \begin{aligned} &y^{hk}(2 + s_{p_1 i_1}) \\ &= \sum_{p \geq p_1;\, q \in \omega} a_{pq}^{hk} + \tfrac{1}{2}[y^{hk}(2 + s_{p_1 i_1} - 2c_{p_1 i_1 1}) + y^{hk}(2 + s_{p_1 i_1} + 2c_{p_1 i_1 1})], \end{aligned}$$

and so there exists $K_0 \in \omega$ such that

$$(2.7) \quad \left| \sum_{p \geq p_1;\, q \in \omega} a_{pq}^{hk} \right| < 4\epsilon \quad \text{for all } k \geqq K_0.$$

Let $z_1 = 2^{-p_0}i_0 + 2^{-p_1-2}$ and let $z_2$ be that integral multiple of $2^{-p_1-2}$ which is closest to $2^{-p_0}(i_0 + 3^{-1})$. From (2.1) it can be verified that there is an interval $M_2$ centered at $z_2$ such that if $t \in M_2$, then $x_{pq}(t)=0$ for all $p$, $q$ such that $p_0 \leqq p \leqq p_1+1$, $q \in \omega$. There is an interval $M_1$ centered at $z_1$ such that for all $t \in M_1$,

$$x_{pq}(t) = \begin{cases} 1 & \text{if } p_0 \leqq p \leqq p_1 - 1 \text{ and } p + q \leqq p_1 - 1. \\ 0 & \text{if } p_0 \leqq p \leqq p_1 + 1 \text{ and } p + q \geqq p_1. \end{cases} \tag{2.8}$$

It may be assumed that $M_1$ and $M_2$ have the same length. Since $M_1 \subseteq I$ and $h \geqq H_0$, there must be a subinterval $M_3$ of $M_1$ such that $|y^h(t)| < 2\epsilon$ for all $t \in M_3$. Since the interval $M_3' = \{t \in M_2 : t-(z_2-z_1) \in M_3\}$ is also contained in $I$, there exists $z \in M_3'$ such that $|y^h(z)| < \epsilon$. If $p \geqq p_1+2$, the function $x_{pq}$ is periodic with period $2^{-p}$ on $I_0$; since $z_2 - z_1$ is an integral multiple of $2^{-p}$, it follows that $x_{pq}(z) = x_{pq}(z-z_2+z_1)$ whenever $p \geqq p_1+2$.

Since $h \geqq H_2$, there is an integer $K_1 \geqq K_0$ such that if $k \geqq K_1$ and $t \in I_0$,

$$\left| \sum_{p<p_0;\, q\in\omega} a_{pq}^{hk} x_{pq}(t) \right| < \epsilon. \tag{2.9}$$

Further, there is a $K_2 \geqq K_1$ such that if $k \geqq K_2$, then $|y^{hk}(z)| < \epsilon$ and $|y^{hk}(z-z_2+z_1)| < 2\epsilon$. Consequently, if $k \geqq K_2$,

$$\begin{aligned} \left| \sum_{p_0 \leq p \leq p_1-1;\, p+q \leq p_1-1} a_{pq}^{hk} \right| = \Big| & y^{hk}(z - z_2 + z_1) - y^{hk}(z) \\ & - \sum_{p<p_0;\, q\in\omega} a_{pq}^{hk} x_{pq}(z - z_2 + z_1) \\ & + \sum_{p<p_0;\, q\in\omega} a_{pq}^{hk} x_{pq}(z) \Big| \\ < 5\epsilon. & \end{aligned} \tag{2.10}$$

Since $h \geqq H_1$ there exists $K_3 \geqq K_2$ such that if $k \geqq K_3$,

$$\left| \sum_{p<p_0;\, q\in\omega} a_{pq}^{hk} \right| = \left| c^{-1} \sum_{p<p_0} y^{hk}(t_{pp}) \right| < c^{-1}(p_0 - 1)\epsilon c p_0^{-1} < \epsilon. \tag{2.11}$$

Since $y^h(1/2) = 1$, there exists $K_4 \geqq K_3$ such that $k \geqq K_4$ implies

$$\left| \sum_{p,q\in\omega} a_{pq}^{hk} - 1 \right| = \left| y^{hk}\left(\frac{1}{2}\right) - 1 \right| < \epsilon. \tag{2.12}$$

Now $y^{hk}(t_{p_0,p_1-1}) = c \sum_{p_0 \leq p \leq p_1-1;\; p+q \geq p_1} a_{pq}^{hk}$, and hence if $k \geqq K_4$ it follows from (2.7), (2.10), (2.11), and (2.12) that

$$
\begin{aligned}
&c^{-1}\left| y^{hk}(t_{p_0,p_1-1}) \right| \\
(2.13)\quad &= \left| \sum_{p,q\in\omega} a_{pq}^{hk} - \sum_{p<p_0;\, q\in\omega} a_{pq}^{hk} - \sum_{p_0\leq p\leq p_1-1;\, p+q\leq p_1-1} a_{pq}^{hk} - \sum_{p\geq p_1;\, q\in\omega} a_{pq}^{hk} \right| \\
&> (1-\epsilon) - \epsilon - 5\epsilon - 4\epsilon = 1 - 11\epsilon.
\end{aligned}
$$

Thus $\|y^{hk}\| > c(1-11\epsilon)$ for all $k \geqq K_4$, and hence $\|y^h\| = \lim_k \|y^{hk}\| \geqq c(1-11\epsilon)$.

We have shown that for every $\epsilon > 0$ there is an $H_2$ such that $\|y^h\| \geqq c(1-11\epsilon)$ for all $h \geqq H_2$. It follows that $\lim \inf_h \|y^h\| \geqq c$.

THEOREM 2. *There exists a real Banach space $X$ such that $K(K(JX))$ is not norm-closed in $X^{**}$.*

PROOF. For each $n \in \omega$ there is, by Theorem 1, a Banach space $X_n$ such that $K_n(K_n(J_nX_n))$ contains an element $x_{on}$ with $\|x_{on}\| = 1$, such that if $\{y_n^h\}_{h\in\omega}$ is a sequence in $K_n(J_nX_n)$ which is $w^*$-convergent to $x_{on}$, then $\lim \inf_h \|y_n^h\| \geqq n^2$. Here $J_n$, for example, is the canonical mapping from $X_n$ into $X_n^{**}$, and $K_n(S)$ is the $w^*$-sequential closure of $S$ in $X_n^{**}$ for each $S \subseteq X_n^{**}$.

Let $X$ be the Banach space $P_{c_0(\omega)}X_n$ as defined in [1, p. 31]. Then $X^*$ is isometrically isomorphic with $P_{l(\omega)}X_n^*$, and $X^{**}$ with $P_{m(\omega)}X_n^{**}$; we shall make the obvious identifications. Now $x_0 = (n^{-1}x_{on})_{n\omega} \in X^{**}$, and indeed $x_0 \in \mathrm{Cl}(K(K(JX)))$, since $x_0$ is the limit in norm of the sequence $\{z_i\}_{i\in\omega}$, where $z_i \equiv (1^{-1}x_{01}, 2^{-1}x_{02}, \cdots, i^{-1}x_{0i}, 0, \cdots) \in K(K(JX))$ for each $i \in \omega$.

If $x_0 \in K(K(JX))$, then $x_0$ would be the $w^*$-limit of a sequence $\{w^h\}_{h\in\omega} \subseteq K(JX)$. For each $n$, $n^{-1}x_{on}$ would be the $w^*$-limit of the sequence $\{w_n^h\}_{h\in\omega} \subseteq K_n(J_nX_n)$; it would then follow that $\lim \inf_h \|w^h\| \geqq \lim \inf_h \|w_n^h\| \geqq n^{-1}n^2 = n$ for each $n \in \omega$, contradicting the fact that a $w^*$-convergent sequence must be bounded in norm. Consequently $x_0 \notin K(K(JX))$, and the proof is complete.

## REFERENCES

1. M. M. Day, *Normed linear spaces*, Springer, Berlin, 1958.

2. P. R. Halmos, *Measure theory*, Van Nostrand, New York, 1950.

3. R. D. McWilliams, *A note on weak sequential convergence*, Pacific J. Math. 12 (1962), 333–335.

4. ———, *On the w*-sequential closure of a cone*, Proc. Amer. Math. Soc. 14 (1963), 191–196.

FLORIDA STATE UNIVERSITY