

# A CONDITION FOR FINITE DIMENSIONAL CONVEXITY

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This note is devoted to a demonstration of

**THEOREM.** *Let  $K$  be a compact contractible subset of  $R^N$ . Suppose every support set is acyclic. Then  $K$  is convex. The converse is obvious.*

Here  $R^N$  is  $N$ -dimensional Euclidean space. We use Čech homology with integer coefficient group [1]. *Plane* will be used for  $N-1$  dimensional hyperplane. A *support plane*,  $P$ , is a plane meeting  $K$  where  $K$  is in just one of the two closed half spaces determined by  $P$ .  $K \cap P$  is the *support set* for  $P$ . A weaker theorem, requiring the added hypothesis of local contractibility for  $K$  was proved by Kuhn [2] who pointed out this hypothesis was essential for any proof based on his methods. Liberman [3] has established a slightly weaker theorem requiring contractibility for the support sets as the culmination of a long chain of subsidiary results.

**PROOF.** We need  $R^n$ , the linear extension of  $K$ , so  $n \leq N$ . To avoid triviality we assume  $n > 1$ . Let  $S^{n-1}$  and  $Y$  refer to the  $n-1$  topological sphere and to the metric unit  $n-1$  sphere respectively. Each point  $y$  in  $Y$  determines a unique supporting hyperplane  $P_y$  whose normals pointing away from  $K$  are parallel translates in  $R^n$  of the vector  $oy$ . We write  $X$  for the set of boundary points of  $K$  and  $X_y$  for the support set  $P_y \cap X$ .

Let

$$\Gamma = \{(y, x) \mid x \in X_y\} \subset Y \times X.$$

For each  $x$  we introduce a subset of  $Y$ ,

$$Y_x = \{y \mid (y, x) \in \Gamma\}.$$

Let  $p$  and  $q$  project  $\Gamma$  into  $Y$  and into  $X$  respectively. Thus

$$p: y \times X_y = y.$$

$$q: Y_x \times x = x.$$

The image of  $\Gamma$  under  $p$  covers  $Y$  while that under  $q$  is  $X' \subset X$ . Obviously  $X$  and  $X'$  are closed in  $R^n$ . Let  $\hat{K}$  be the convex hull of  $K$  and denote its boundary by  $\hat{K}^\bullet$ . It is important to note that:

$$(a) \quad X' \subset \hat{K}^\bullet.$$

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Observe that  $Y_x$  is not affected if  $X'$  is replaced by  $\hat{K}$ . This is to say the join  $OY_x$  of  $O$  and  $Y_x$  is the translation to origin  $O$  of the normal cone to  $\hat{K}$  at  $x$ . The set  $Y_0 \subset Y$  is said to be sphere convex if every pair of distinct points in  $Y_0$  are endpoints of a unique great circle arc contained in  $Y_0$ . It is well known that the normal cone at a boundary point of a convex set is closed and convex. Hence  $OY_x$  is closed convex. The use of  $R^n$  (in place of  $R^N$ ) assures the absence of antipodal points in  $Y_x$ . Accordingly  $Y_x$  is closed and sphere convex. It is therefore obviously contractible over  $Y$  and so acyclic. (If the  $N-1$  unit sphere is used for  $Y$ , where  $N > n$ , then  $Y_x$  need not be acyclic.)

Both  $p$  and  $q$  are continuous and  $p^{-1}y$  is homeomorphic to the acyclic support set  $X_y$  while  $q^{-1}x$ , for  $x \in X'$ , is homeomorphic to the acyclic set  $Y_x$ . The Vietoris-Begle Theorem [1, p. 503] guarantees therefore that

$$(b) \quad H_*(X') \approx H_*(\Gamma) \approx H_*(Y).$$

We remark also that  $\hat{K} \cdot$  is of course an  $S^{n-1}$ . Thus

$$(c) \quad H_*(Y) \approx H_*(\hat{K} \cdot) \approx H(S^{n-1}).$$

From (b) and (c) results

$$H_*(X') \approx H_*(S^{n-1}).$$

However since a (closed) proper subset of a topological  $n-1$  sphere has a trivial  $n-1$  homology group, it follows from (a) that

$$X' = \hat{K} \cdot.$$

A Euclidean  $n-1$  sphere,  $n > 1$ , is not contractible over a proper subset of the disk it bounds. Accordingly, if  $X'$ , the homeomorph of such a sphere is a proper subset of  $X$ , then  $K$  is a proper subset of  $\hat{K}$  and so  $X'$  is not contractible over  $K$  and a fortiori neither is  $K$  in contradiction with our hypothesis.

One may conjecture that a similar theorem is valid in a Banach space.

#### BIBLIOGRAPHY

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