CONTRACTIBLE COMPLEXES IN S^n

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1. **Introduction.** By a pseudo n-cell is meant a contractible compact combinatorial n-manifold with boundary. Poenaru [9] and Mazur [7] gave the first examples of pseudo 4-cells which are not topological 4-cells, but whose products with the unit interval are topologically 5-cells. Newman [8] defines a 2-complex P such that $\pi_1(P)\neq 1$, while $H_1(P, Z)=0=H_2(P, Z)$. Curtis [4] making use of this 2-complex has shown that, for each $n\geqq 4$, there exists a pseudo n-cell which is not a topological n-cell because its boundary fails to be simply connected. Curtis [4] also shows that the cartesian product of a pseudo n-cell and an interval is the topological $(n+1)$-cell, I^{n+1} if $n\geqq 5$.

Curtis [5] making use of Mazur's peculiar embedding of the dunce hat in S^4 [7], [13] gives an example of a contractible 2-complex K embedded as a subcomplex of a combinatorial triangulation of S^4 such that $\pi_1(S^4-K)\neq 1$. The purpose of this paper is to show that for $n\geqq 4$ there exists a contractible $(n-2)$-complex K^{n-2} combinatorially embedded in S^n such that $\pi_1(S^n-K^{n-2})\neq 1$. The regular neighborhood $N^n=N(K^{n-2})$ of K^{n-2} in S^n is also a pseudo n-cell which fails to be a topological n-cell and its product with the unit interval I is shown to be a combinatorial $(n+1)$-cell, rather than just merely topological. In addition, each N^n $(n\geqq 5)$ gives examples of combinatorial n-manifolds with boundary which are not topologically I^n but can be expressed as the union of two combinatorial n-balls whose intersection is also a combinatorial n-ball.

2. **Definitions.** We will use the terminology of [12], [13]. All manifolds and all mappings or homeomorphisms will be considered in the combinatorial sense. We will use $\approx$ to denote combinatorial equivalence. If the complex K collapses to the complex L, this will be denoted $K\searrow L$.

Let $f: X\rightarrow Y$ be continuous. The identification space Y_f derived from $(X\times[0, 1])\cup Y$ by identifying each point $(x, 1)$ with the point

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$f(x)$ in Y and using the identification topology is called the mapping cylinder of f.

3. **Preliminaries.** The following two lemmas are well known and elementary, hence no proof will be included.

LEMMA 1. *If C is a k-complex embedded as a finite subcomplex of a combinatorial n-sphere S^n and M is a regular neighborhood of C in S, with $C\subset \operatorname{int} M$, then there is a combinatorial map ϕ: $\operatorname{Bd} M\to C$ such that M is combinatorially equivalent to $I\times \operatorname{Bd} M\cup_\phi C$, the mapping cylinder of ϕ.*

LEMMA 2. *Suppose C is a k-complex embedded as a finite subcomplex of a combinatorial n-sphere S^n and N is any regular neighborhood of C in S^n, such that $C\subset \operatorname{int} N$; then $\pi_1(N-C)=\pi_1(\operatorname{Bd} N)$.*

The topological dunce hat D is obtained from a triangle abc say, by identifying all three sides $ab=bc=ac$.

THEOREM 1. *There exist two combinatorially inequivalent embeddings D_1, D_2 of the dunce hat D in S^4, such that the regular neighborhood N_1 of D_1 is combinatorially I^4, while N_2 the regular neighborhood of D_2 is not topologically I^4. Moreover, $\pi_1(\operatorname{Bd} N_2)\neq 1$, $\pi_1(N_2-D_2)\neq 1$, but $\pi_1(S^4-D_2)=1$.*

PROOF OF THEOREM 1. Let D_1 be any combinatorial embedding of D in $S^3\subset S^4$. Then $N_1\searrow \hat{N}_1$, the regular neighborhood of D in S^3, and since $\hat{N}_1\approx I^3$, $N_1\approx I^4$.

For D_2 we will use Mazur's embedding of D in S^4 (as in Theorem 5 [**13**]). Since $N_2\approx W^4$ (also Theorem 5 [**13**]) and $\pi_1(\operatorname{Bd} W^4)\neq 1$ (see [**7**]) we have that $N_2\not\approx I^4$. The fact that $\pi_1(N_2-D_2)\neq 1$ follows from Lemma 2. We see that $\pi_1(S^4-D_2)=1$ by considering Mazur's embedding of D in S^4. That is $D\subset W^4\subset 2W^4\approx S^4$. Since $S^4-D_2\approx W^4\cup W^4-D_2\approx W^4\cup(\operatorname{Bd} W\times[0, 1))$ (using Lemma 1), we see that S^4-D_2 is of the same homotopy type as W^4 and $\pi_1(S^4-D_2)=1$.

To see that these two embeddings are combinatorially inequivalent, suppose there exists a p.w.l. homeomorphism taking S^4 onto S^4 carrying D_1 onto D_2. Let a_1, a_2 be the points of D_1, D_2 respectively, which correspond to the point $a(=b=c)$ in D. Then by subdividing the triangulation of S^4 so that h is simplicial, we get that h carries $\operatorname{st}(a_1, S^4)$ onto $\operatorname{st}(a_2, S^4)$, each combinatorial 4-balls. Also h carries $\operatorname{lk}(a_1, D_1)\subset\operatorname{lk}(a_1, S^4)\approx S^3$ onto $\operatorname{lk}(a_2, D_2)\subset\operatorname{lk}(a_2, S^4)\approx S^3$. This leads to a contradiction, since there exists no homeomorphism of S^3 onto S^3 carrying $\operatorname{lk}(a_1, D_1)$ as in $\operatorname{lk}(a_1, S^4)$ onto $\operatorname{lk}(a_2, D_2)$ as in $\operatorname{lk}(a_2, S^4)$. See Figures 5 and 8 of [**13**].

THEOREM 2. *There exists a contractible* 2*-complex* K *and two inequivalent embeddings* K_1, K_2 *of* K *in* S^4 *so that the regular neighborhood* N_1 *of* K_1 *is a combinatorial* 4*-ball, while* $\pi_1(S^4-K_2)\neq 1$.

REMARK. Since $N_1\approx I^4$, K_1 is cellular in S^4 and hence $S^4-K_1=E^4$ and $\pi_1(S^4-K_1)=1$. Also it will follow from a later result, which does not use the particular construction of the embedding of K_2 in S^4, that if N_2 is the regular neighborhood of K_2 in S^4 then $\pi_1(\text{Bd } N_2)\neq 1$ and hence $N_2\neq I^4$.

PROOF OF THEOREM 2. K will be the union of two disjoint copies of the dunce hat D joined together by a polyhedral segment intersecting each in $a(=b=c)$. K_1 will be the embedding of K in $S^3\subset S^4$ and $N_1\approx I^4$ as in Theorem 1.

To get K_2, we will use Curtis's modification [5]. Let us again consider S^4 as $2W^4$ (Mazur's pseudo 4-cell). We have a D' and D'' (copies of D) in each copy of W^4. Since $S^4-(D'+D'')\approx(W^4-D')\cup(W^4-D'')\approx(\text{Bd } W^4\times[0,\ 1))\cup(\text{Bd } W^4\times[0,\ 1))$ and $\pi_1(\text{Bd } W^4)\neq 1$, we have $\pi_1(S^4-(D'+D''))\neq 1$. Let A be a polyhedral arc in S^4 such that $A\cap D'=a'$, $A\cap D''=a''$ (where a', a'' correspond to $a(=b=c)$ in D) and $A\cap\text{Bd } W^4=\{p\}$. Such an A can easily be gotten because of the particular embedding of D', D'' in each copy of W^4. Then $K_2=D'\cup A\cup D''$ will be an embedding of K in S^4 such that $\pi_1(S^4-K_2)\neq 1$.

Finally, it is clear that the embeddings of K_1 and K_2 in S^4 are inequivalent since the fundamental groups of their complements are different.

THEOREM 3. *If* N_2 *is the regular neighborhood of* K_2 *in* S^4 *then* $N_2\times I\approx I^5$.

PROOF OF THEOREM 3. Since $K_2\approx D\cup A\cup D$, two disjoint copies of D joined together by a polyhedra arc intersecting each D in the point a and $D\times I\searrow\{a\}$ (Theorem 1 [13]), it follows easily that $K_2\times I\searrow 0$. Hence $N_2\times I\searrow K_2\times I\searrow 0$ and this implies that $N_2\times I$ is a combinatorial 5-ball (Corollary 1_n [12]).

THEOREM 4. *Suppose* K *is a contractible* 2*-complex such that* $K\times I\searrow 0$ *and* K *is embedded in the interior of a contractible* 4*-manifold with boundary* $W^4\subset E^4$ *such that* $\pi_1(W^4-K)\neq 1$. *Then given any combinatorial triangulation* T *of* E^4 *which contains* K *as a subcomplex, there exists no* 3*-manifold* (*with or without boundary*) *in* E^4 *which is a subcomplex of* T *containing* K.

REMARK. Mazur's embedding of D in S^4 is such a contractible 2-complex. It follows from the theorem that even though D can be embedded in E^3, for this particular embedding it lies in no 3-manifold in E^4.

PROOF OF THEOREM 4. Suppose there exists such a 3-manifold M^3, that is $K \subset M^3 \subset T$. Then for some subdivision of T and hence of M^3, say $\hat{T}$, we would have $N(K, M^3) \subset \text{int } W^4$, where $N(K, M^3)$ denotes the simplicial neighborhood of K in M^3 under the second barycentric subdivision of $\hat{T}(M^3)$. Also let us suppose that $\hat{T}$ is so fine that $N(N(K, M^3), \hat{T}) \subset \text{int } W^4$. Now if $N(K, M^3) = I^3$, then $N(N(K, M^3), \hat{T}) = I^4 \subset \text{int } W^4$. We then could use Bd $I^4 = S^3$ to shrink nontrivial curves of $W^4 - K$ missing K. (Also see Theorem 6 of [13].) Therefore, $N(K, M^3) \neq I^3$. However, $N(K, M^3) \times I \searrow K \times I \searrow 0$ and this implies that $N \times I = I^4$ which in turn implies $N = I^3$ ([1], [2]) which contradicts the above. This contradiction arose by assuming there existed an M^3 with $K \subset M^3 \subset T$.

4. **Contractible complexes in S^n.** If K is a k-complex of a combinatorial n-sphere S^n, we will use $N(K, S^n)$ to denote the canonical regular neighborhood of K under the second barycentric subdivision of S^n. ΣK and CK will denote the suspension of K and cone over K respectively. Also, we will write $\Sigma K = C^+K \cup C^-K$ with $C^+K \cap C^-K = K$, where in letting p and q denote the "top" and "bottom" points of ΣK used in getting the suspension of K, we have that C^+K is the cone over K in ΣK from p and C^-K is the cone over K in ΣK from q.

LEMMA 3. *Suppose $\hat{K}$ is a k-complex in S^n such that $N(\hat{K}, S^n) \approx I^n$ and B^n is a combinatorial n-ball in S^n such that $N(\hat{K}, S^n) \subset \text{int } B^n$. If $\Sigma \hat{K} \equiv K$ is considered as embedded in $S^{n+1} \equiv B^{n+1} \cup C(\text{Bd } B^{n+1})$, where $B^{n+1} \equiv \Sigma B^n$, then $N(K, S^{n+1}) \approx I^{n+1}$ and $\pi_1(S^{n+1} - K) = 1$.*

PROOF OF LEMMA 3. $\Sigma[N(\hat{K}, S^n)]$ is a regular neighborhood of K in S^{n+1}. That is, $\Sigma[N(\hat{K}, S^n)] \searrow \Sigma \hat{K} \equiv K$ since $N(\hat{K}, S^n) \searrow \hat{K}$ and it is an n-manifold with boundary since $\Sigma I^n \approx I^{n+1}$. Hence $I^{n+1} \approx \Sigma[N(\hat{K}, S^n)] \approx N(K, S^{n+1})$ (Theorem 23_n [12]). It follows that $\pi_1(S^{n+1} - K) = 1$ since K is cellular in S^{n+1} (the decreasing sequence of $(n+1)$-cells are the canonical regular neighborhoods of K under increasingly higher order barycentric subdivisions of S^{n+1}). That is $S^{n+1} - K = E^{n+1}$ [2].

LEMMA 4. *Suppose $\hat{K}$ is a k-complex in S^n $(n \geqq 3)$ such that $\pi_1(S^n - \hat{K}) \neq 1$ and B^n is a combinatorial n-ball in S^n such that $\hat{K} \subset \text{int } B^n$. Then if $\Sigma \hat{K} \equiv K$ is considered as embedded in S^{n+1} as in Lemma 3, then $\pi_1(S^{n+1} - K) \neq 1$.*

PROOF OF LEMMA 4. Since $\pi_1(S^n - \hat{K}) \neq 1$, we have that $\pi_1(B^n - \hat{K}) \neq 1$. Also $\Sigma B^n - \Sigma \hat{K} \equiv B^{n+1} - K \approx (B^n - \hat{K}) \times (-1, 1)$. Hence

$\pi_1(B^{n+1}-K)\neq 1$. The claim is that $\pi_1(S^{n+1}-k)\neq 1$. Suppose otherwise. Let J be any polyhedral simple closed curve in $B^{n+1}-K$ which is nontrivial in $B^{n+1}-K$. Suppose J bounds a polyhedral singular disk D in $S^{n+1}-K$. Let p, q be the suspension points of ΣB^n and r the vertex point in $C(\mathrm{Bd}(\Sigma B^n))$. Since $n+1\geqq 4$, we can adjust D to a singular disk D' (keeping J fixed) so that $D'\cap$(polyhedral arc prq) $=\emptyset$. But then D' can be retracted onto a singular disk D'' bounded by J in $B^{n+1}-K$ by projecting the part of D' not in B^{n+1} from r onto $\mathrm{Bd}\, B^{n+1}-\{p+q\}$. This leads to a contradiction that $\pi_1(B^{n+1}-K)\neq 1$, therefore $\pi_1(S^{n+1}-K)\neq 1$.

LEMMA 5. *If K is a k-complex in S^n and $\pi_1(S^n-K)\neq 1$, denoting $N(K, S^n)$ by N, then $N\neq I^n$, $\pi_1(N-K)=\pi_1(\mathrm{Bd}\, N)\neq 1$ and $\pi_1(\mathrm{Cl}(S^n-N))\neq 1$.*

PROOF OF LEMMA 5. If $N=I^n$ then K is cellular in S^n and this would imply that $\pi_1(S^n-K)=1$, contradicting the hypothesis of the lemma. Also, $S^n-K\approx(N-K)\cup\mathrm{Cl}(S^n-N)\approx([0,1)\times\mathrm{Bd}\, N)\cup\mathrm{Cl}(S^n-N)$ (by Lemma 1). Hence S^n-K is homotopically equivalent to $\mathrm{Cl}(S^n-N)$. Therefore $\pi_1(\mathrm{Cl}(S^n-N))\neq 1$.

Suppose $\pi_1(\mathrm{Bd}\, N)=1$. Since $S^n=N\cup\mathrm{Cl}(S^n-N)$ and $N\cap\mathrm{Cl}(S^n-N)=\mathrm{Bd}\, N$, if $\pi_1(\mathrm{Bd}\, N)=1$, then using van Kampen's theorem we get that $\pi_1(S^n)$ is the free product $\pi_1(N) * \pi_1(S^n-N)$, which would not be trivial (Corollary 6.4.5, p. 244, [6]). Therefore, $\pi_1(\mathrm{Bd}\, N)\neq 1$ and by Lemma 2 $\pi_1(\mathrm{Bd}\, N)=\pi_1(N-K)\neq 1$.

LEMMA 6. *Suppose $\hat{K}$ is a k-complex in S^n such that $\hat{K}\times I\searrow 0$. Let $K\equiv\Sigma\hat{K}$ be p.w.l. embedded in S^{n+1} (not necessarily as in Lemma 3), then $N(K, S^{n+1})\times I\approx I^{n+2}$.*

PROOF OF LEMMA 6. First we note that if $\hat{L}$ is a subcomplex of $\hat{K}$ such that $\hat{K}\searrow\hat{L}$, if $K\equiv\Sigma\hat{K}$ and $L\equiv\Sigma\hat{L}$ then $K\searrow L$. This follows by induction on the number of simplexes of $\hat{K}-\hat{L}$. Next we observe that if $\hat{K}$ is a complex such that $\hat{K}\searrow 0$ then $K\equiv\Sigma\hat{K}\searrow 0$. This follows since $\hat{K}\searrow\{v\}$ (v some vertex of $\hat{K}$) and by the above remark $K\equiv\Sigma\hat{K}\searrow\Sigma v\searrow v$. Finally, if $\hat{K}$ is a complex such that $\hat{K}\times I\searrow 0$ and if $K\equiv\Sigma\hat{K}$, then $K\times I\searrow 0$. This follows since $\Sigma\hat{K}\times I\searrow\Sigma(\hat{K}\times I)$ and $\Sigma(\hat{K}\times I)\searrow 0$ by the second remark.

Therefore since $\hat{K}\times I\searrow 0$, we have that $K\times I\searrow 0$. Hence, $N(K, S^{n+1})\times I\searrow K\times I\searrow 0$ and $N(K, S^{n+1})\times I\approx I^{n+2}$.

THEOREM 5. *For $n\geqq 4$ there exists a contractible $(n-2)$-complex P and two inequivalent embeddings P_1, P_2 of P in S^n such that the regular neighborhood N_1 of P_1 is a combinatorial n-ball and $\pi_1(S^n-P_1)=1$. However, $\pi_1(S^n-P_2)\neq 1$ and if N_2 is the regular neighborhood of P_2,*

$N_2 \not\approx I^n$, $\pi_1(\text{Bd } N_2) = \pi_1(N_2 - P_2) \neq 1$ *and* $N_2 \times I$ *is a combinatorial* $(n+1)$*-ball. Moreover* $P \times I \searrow 0$.

PROOF OF THEOREM 5. The proof will be by induction. For $n=4$ the result follows from Theorem 2, Lemma 5, and Theorem 3. Suppose inductively for $n=k$ we have a contractible $(k-2)$-complex p^{k-2}, two embeddings P_1^{k-2}, P_2^{k-2} in S^k such that $N_1^k \approx I^k$ and $\pi_1(S^k - P_1^{k-2}) = 1$, while $\pi_1(S^k - P_2^{k-2}) \neq 1$, $N_2^k \not\approx I^k$, $\pi_1(\text{Bd } N_2^k) = \pi_1(N_2^k - P_2^{k-2}) \neq 1$ and $N_2^k \times I \approx I^{k+1}$. Also assume $P^{k-2} \times I \searrow 0$.

Using Lemma 3 we get a contractible $(k-1)$-complex $P_1^{k-1} \approx \Sigma P_1^{k-2}$ in S^{k+1} such that $N(P_1^{k-1}, S^{k+1}) \approx I^{k+1}$ and $\pi_1(S^{k+1} - P_1^{k-1}) = 1$. Using Lemma 4 we get a contractible $(k-1)$-complex $P_2^{k-1} \equiv \Sigma P_2^{k-2}$ in S^{k+1} such that $\pi_1(S^{k+1} - P_2^{k-1}) \neq 1$. Lemma 5 then implies that $N_2^{k+1} \not\approx I^{k+1}$, $\pi_1(N_2^{k+1} - P_2^{k-1}) = \pi_1(\text{Bd } N_2^{k+1}) \neq 1$. Since $P_2^{k-1} \equiv \Sigma P_2^{k-2}$ and $P_2^{k-2} \times I \searrow 0$, the third remark in the proof of Lemma 6 gives us that $P_2^{k-1} \times I \searrow 0$. Also, Lemma 6 gives us that $N(P_2^{k-1}, S^{k+1}) \times I \approx I^{k+2}$. Finally, since $P_1^{k-2} \approx P_2^{k-2}$ and $P_i^{k-1} \equiv \Sigma P_i^{k-2}$ $(i=1, 2)$ we have that $P_1^{k-1} \approx P_2^{k-2}$.

COROLLARY 6. *For* $n \geqq 4$ *there exists a contractible* $(n-1)$*-complex* K^{n-1} *in* S^n *such that* $N(K, S^n) \not\approx I^n$, $\pi_1(\text{Bd } N(K, S^n)) \neq 1$ *and* $N(K, S^n) \times I \approx I^{n+1}$. *Also* $\pi_1(S^n - K^{n-1}) \neq 1$.

COROLLARY 7. *For* $n \geqq 4$ *there exists a contractible n-complex* (*combinatorial n-manifold with boundary*) N^n *in* S^n *such that* $N^n \not\approx I^n$, $\pi_1(\text{Bd } N^n) \neq 1$ *and* $N^n \times I \approx I^{n+1}$. *Also* $\pi_1(S^n - N^n) \neq 1$.

Corollary 7 follows from Theorem 5 by taking $N^n = N_2$ of that theorem; Corollary 6 by reducing N^n to K^{n-1} using Whitehead elementary contractions and the fact that $N(K^{n-1}, S^n) \approx N^n$. $\pi_1(S^n - N^n) \neq 1$ since $\pi_1(\text{Cl}(S^n - N_2)) \neq 1$ by Lemma 5. $\pi_1(S^n - K^{n-1}) \neq 1$ since we can assume that $K^{n-1} \subset \text{int } N^n$ and hence $S^n - K^{n-1}$ is of the same homotopy type as $\text{Cl}(S^n - N^n)$ (using Lemma 1).

THEOREM 6. *For* $n \geqq 5$, N_2^n (*of Theorem* 5) *is a contractible combinatorial n-manifold with boundary which is not topological* I^n, *but is combinatorially equivalent to the union of two combinatorial n-balls*, $B_1^n \cup B_2^n$ *such that* $B_1^n \cap B_2^n \approx B_3^n$ *a combinatorial n-ball which is a subcomplex of each. Furthermore*, $\text{int } N_2^n \approx X \cup Y$ *where* $X \approx Y \approx X \cap Y \approx E^n$, *while* $\text{int } N_2^n \not\approx E^n$.

PROOF OF THEOREM 6. For $n \geqq 5$, $N_2^n \approx N(P_2^{n-2}, S^n) \approx N(\Sigma P_2^{n-3}, S^n)$. Also, $N_2^{n-1} = N(P_2^{n-3}, S^{n-1}) \not\approx I^{n-1}$ with $N_2^{n-2} \times I \approx I^n$. We observe that $N_2^n \approx N(C^+ P_2^{n-3}, S^n) \cup N(C^- P_2^{n-3}, S^n)$. Since $C^+ P_2^{n-3} \searrow 0$ and $C^- P_2^{n-3} \searrow 0$, Theorem 23_n [12] gives us that $N(C^+ P_2^{n-3}, S^n) \approx B_1^n$ and

$N(C^-P_2^{n-3}, S^n)\approx B_2^n$, the two desired combinatorial n-balls. Since S^n was obtained as $\Sigma B^{n-1}\cup C(\mathrm{Bd}\,\Sigma B^{n-1})$ where $N_2^{n-1}=N(P_2^{n-3}, S^{n-1})$ lies in int B^{n-1}, for some $B^{n-1}\subset S^{n-1}$, it follows that $B_1^n\cap B_2^n$ $\approx N(C^+P_2^{n-3}, S^n)\cap N(C^-P_2^{n-3}, S^n)\approx N(P_2^{n-3}, S^n)\approx N(P_2^{n-3}, S^{n-1})\times I$ which is $\approx I^n$, that is, our B_3^n. $N(P_2^{n-3}, S^n)\approx N(P_2^{n-3}, S^{n-1})\times I$ since the latter expression is clearly a regular neighborhood of P_2^{n-3} in S^n and any two regular neighborhoods of the same complex are combinatorially equivalent.

Letting $X=\mathrm{int}\,B_1^n$, $Y=\mathrm{int}\,B_2^n$, then $X\cap Y\approx\mathrm{int}\,B_3^n$ so that int N_2^n $\approx X\cup Y$ and $X\approx Y\approx X\cap Y\approx E^n$. We have that int $N_2^n\not\approx E^n$ since N_2^n is an n-manifold with boundary (and hence collared from the inside [3]) and $\pi_1(\mathrm{Bd}\,N_2^n)\neq 1$. That is, if int $N_2^n=E^n$, simple closed curves near "infinity" can be shrunk near "infinity," but Bd $N_2^n\times[0, 1)$ the collar of Bd N_2^n in N_2^n is not simply connected and hence there exist nontrivial simple closed curves in Bd $N_2^n\times(0, 1)$.

THEOREM 7. *Suppose C' is a contractible k-complex that can be p.w.l. embedded in a combinatorial n-sphere S^n with triangulation T as a subcomplex C such that $\pi_1(S^n-C)\neq 1$ (necessarily $k=n$, $n-1$, or $n-2$ by Lemma 5 and Theorem 1 [5]). Then for $n\geqq 5$ there exists a p.w.l. embedding $\hat{C}$ of C' in S^n under T such that $N(\hat{C}, S^n)$ $\approx N(C, S^n)(\not\approx I^n)$, but now $\pi_1(S^n-\hat{C})=1$.*

PROOF OF THEOREM 7. Let Σ be the combinatorial n-manifold formed by attaching two copies of $N(C, S^n)$ together along their boundaries. Since $N(C, S^n)$ is contractible, Σ is a combinatorial n-manifold with the homotopy type of S^n. Hence for $n\geqq 5$, Σ is a topological n-sphere which is also a combinatorial n-manifold [10], [14]. Let us also denote C in Σ as the complex C in one copy of $N(C, S^n)$ used in forming Σ. Now $\pi_1(\Sigma - C) = 1$ since $\Sigma - C = \{[0, 1)\times\mathrm{Bd}\,N(C, S^n)\}\cup N(C, S^n)$ which is homotopically equivalent to $N(C, S^n)$. Let p be an interior point of some n-simplex of Σ missing the copy of $N(C, S^n)$ in Σ containing C. Now $\Sigma-\{p\}$ is p.w.l. equivalent to $S^n-\{q\}$ under T for some $q\in S^n$ since $n\geqq 5$ [11]. Hence there exists a p.w.l. homeomorphism h of $\Sigma-\{p\}$ onto $S^n-\{q\}$ taking C and $N(C, S^n)$ (as in $\Sigma-\{p\}$) into $S^n-\{q\}$ (under T). Then $h(C)=\hat{C}$ is a p.w.l. embedding of C' in S^n and $\pi_1(S^n-\hat{C})=1$ since $\pi_1(\Sigma-C)=1$. Since $h(N(C, S^n))$ is a regular neighborhood of $\hat{C}$ in S^n under a subdivision of T, $N(\hat{C}, S^n)\approx h(N(C, S^n))\approx N(C, S^n)$. Note, if C' is the contractible k-complex given in Theorem 5, then one has that Σ is in fact a combinatorial n-sphere (since $N\times I\approx I^{n+1}$). Hence $\Sigma\approx S^n$ under T and the result follows immediately.

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