SOME REMARKS ON BOUNDARY VALUE PROBLEMS[1]

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Schechter [6] considers the problem of finding necessary and sufficient conditions for the existence of solutions in various spaces to general boundary value problems for an arbitrary partial differential operator A. The boundary conditions are used to determine certain subspace $V^{s,p}$ of $H^{s,p}$. The conditions are stated in terms of certain inequalities under conditions which are satisfied by general elliptic boundary value problems for regions Ω relatively compact in R^n. The main tool is a representation theorem for continuous linear functionals on certain subspaces of $H^{s,p}$. The basic assumption is that the kernel N' of the adjoint operator A' is finite dimensional.

If Ω is relatively compact the a priori inequalities as proved by Agmon, Douglis and Nirenberg [1]; Schechter [5]; and Browder [2] for example applied to A' together with Rellich's lemma yield the finite dimensionality of N'. The a priori estimates are true on regions Ω which are not necessarily relatively compact but then we do not know N' is finite dimensional. It seems of interest therefore to know that at least in the Hilbert space setting Schechter's results are true even if N' is not finite dimensional. The a priori inequality tells us that on N', $H^{2m,p}$ and $H^{0,p}$ induce the same topology. In what follows we show that when $p=2$, this is all we need to know to obtain Schechter's representation theorem. These results can be stated abstractly and we do so here.

If E and F are two topological vector spaces we use the notation $E \subset F$ to mean that (i) E is a subset of F and (ii) the canonical injection of E into F is continuous, i.e., that E has a finer topology than F. We use the term E is dense in F to mean that E with topology induced on it by F is dense in F. Finally we use the notation $\mathcal{L}(E, F)$ for the set of continuous linear maps of E into F.

In what follows H^0 and H' will be Hilbert spaces with $H' \subset H^0$, H' dense in H^0. For convenience we will suppose that $\|u\|_0 \leqq \|u\|_1$ for $u \in H'$. N will be a closed subspace of H' for which there exists a constant c_0 such that $\|u\|_1 \leqq c_0 \|u\|_0$ for $u \in N$. Thus N is also a closed subspace of H^0 and H^0 and H' induce the same topology on N.

1. THEOREM. *There is a positive-definite, self-adjoint operator B with domain equal to H' such that for $u \in H'$, $\|u\|_1 = \|Bu\|_0$.*

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This follows from a well-known result of Friedrichs [4] and the Spectral Theorem.

Again using the Spectral Theorem for $0<t<1$ we let H^t be the domain of B^t and for $u\in H^t$ we set $\|u\|_t=\|B^t u\|_0$. The H^t are called interpolation spaces.

2. Proposition. *With the norms* $\|\cdot\|_t$ *the* H^t *for* $0<t<1$ *are Hilbert spaces and for* $0<s<t<1$, $H^t\subset H^s$ *with* H^t *dense in* H^s.

3. Theorem (Lions [3]). *Let* K^0 *and* K' *be another pair of Hilbert spaces with* $K'\subset K^0$ *and* K' *dense in* K^0. *Let* K^t, $0<t<1$ *be the analogous interpolation spaces. If* $T\in\mathcal{L}(H^0, K^0)$ *and* $T\in\mathcal{L}(H', K')$ *then for* $0<t<1$, $T\in\mathcal{L}(H^t, K^t)$.

4. Definition.[2] For $u\in H^0$ let

$$\|u\|_{-t} = \sup\{\,|(u, v)| : v\in H^t \text{ and } \|v\|_t \leq 1\}.$$

Let H^{-t} be the completion of H^0 in the norm $\|\cdot\|_{-t}$.

5. Theorem. *There is a canonical topological isomorphism between* H^{-t} *and the dual space of* H^t *for* $0\leq t\leq 1$.

6. Lemma. *For* $u\in N$, $\|u\|_0\leq c_0\|u\|_{-1}$.

Proof. Using the Projection Theorem we write $u=u'+u''$, with $u'\in N$ and $u''\in N_\perp=\{v\in H^0: (u, v)=0 \text{ for } u\in N\}$. Then

$$\begin{aligned}\|u\|_0 &= \sup\{\,|(u, v)| : \|v\|_0 \leq 1\} = \sup\{\,|(u, v)| : v\in N, \|v\|_0\leq 1\}\\ &\leq c_0 \sup\{\,|(u, v)| : v\in N, \|v\|_1\leq 1\}\\ &\leq c_0 \sup\{\,|(u, v)| : \|v\|_1 \leq 1\} = c_0\|u\|_{-1}.\end{aligned}$$

For $-1\leq t\leq 1$ we let N^t denote N with the topology induced by H^t. By the preceding lemma and Theorem 3 we have

7. Proposition. *If* $-1\leq s, t\leq 1$ *the spaces* N^s *and* N^t *are topologically isomorphic.*

Using the notation of the proof of Lemma 6 the map $P: H^0\to N$ given by $u\to u'$ is continuous.

8. Proposition. $P\in\mathcal{L}(H^t, N^t)$ *for* $-1\leq t\leq 1$.

Proof. We let $Pu=u'$. Then for $u\in H'$ $\|u'\|_1\leq c_0\|u'\|_0\leq c_0\|u\|_0 \leq c_0\|u\|_1$.

Now for $u\in H^0$, $\|u'\|_{-1}^2\leq\|u'\|_0^2=|(u', u')|=|(u, u')|\leq\|u\|_{-1}\|u'\|_1 \leq c_0^2\|u\|_{-1}\|u'\|_{-1}$. Thus $\|u'\|_{-1}\leq c_0^2\|u\|_{-1}$. For $-1<t<0$ and $0<t<1$, apply Theorem 3.

[2] We use $(\cdot\,,\cdot)$ instead of $(\cdot\,,\cdot)_0$ for the scalar product in H^0.

Let $N_\perp^t$ for $-1 \leqq t \leqq 1$ be the set of $w \in H^t$ such that for $u \in N$, $(u, w) = 0$. The following two results are then simple consequences of the preceding facts.

9. THEOREM. *Let* $u \in H^t$, $-1 \leqq t \leqq 1$. *Then* $u = u' + u''$ *with* $u' \in N$ *and* $u'' \in N_\perp^t$.

10. LEMMA. *If* $-1 \leqq t \leqq 1$ *and* $v \in N_\perp^t$ *then*

$$\|v\|_t \leqq c_t \sup\{ |(u, v)| : u \in N_\perp^{-t} \text{ and } \|u\|_{-t} \leqq 1\}.$$

11. THEOREM. *For* $-1 \leqq t \leqq 1$ *let* f *be a continuous linear functional on* $N_\perp^t$. *Then there exists a* $v \in N_\perp^{-t}$: $f(u) = (u, v)$ *for* $u \in N_\perp^t$.

Using the preceding results the proof is identical to that given in Schechter [6].

12. REMARK. Let V be a closed subspace of H' containing N. For $u \in H^0$ let $\|u\|_{V,-1} = \sup\{ |(u, v)| : v \in V \text{ and } \|v\|_1 \leqq 1\}$. Let V^{-1} be the completion of H^0 in the norm $\|\cdot\|_{V,-1}$. Clearly $H^{-1} \subset V^{-1}$. It is easy to see that V^{-1} can be identified with the dual space of V and that Proposition 8 is true for V^{-1}.

13. REMARK. Using Propositions 7 and 8 it is not hard to show that the estimates of Schechter [7] and the L^2 version of the estimates of Schechter [8], [9] can be obtained without assuming the finite dimensionality of kernel of the elliptic operator. The closure of the image in L^2 seems to be essential.

Added in proof. The L^2 version of the estimates of Schechter, Math. Scand. (1963), 47–69, can also be obtained from these results.

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