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A REMARK ON WIENER'S TAUBERIAN THEOREM

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A recent note by Levinson [1] made it seem worthwhile to point out that a weaker version of the Tauberian theorem can be proved in a few lines which is, however, strong enough to provide a proof of the prime number theorem.

Let $K(x)\in L(-\infty,\infty)$ and assume that its Fourier transform obeys the standard condition

$$\kappa(\xi) = \int_{-\infty}^{\infty} K(x)e^{i\xi x}\,dx \neq 0 \quad \text{for all } -\infty < \xi < \infty. \tag{1}$$

One version of Wiener's Tauberian theorem is the assertion that if $m(y)$ is a bounded measurable function such that for almost all x,

$$\int_{-\infty}^{\infty} K(x-y)m(y)\,dy = 0 \tag{2}$$

then $m(y)=0$ almost everywhere.

The weaker version of the Tauberian theorem is obtained by adding an extra requirement on the function $K(x)$, namely that

$$x^2K(x) \in L(-\infty,\infty). \tag{3}$$

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To use this to prove the prime number theorem, we can follow the proof given by Levinson, since here one had

$$K(x) = \begin{cases} 0 & \text{for } x \leqq 0 \\ R(e^x)e^{-x} & \text{for } x > 0 \end{cases}$$

where R is a bounded function; condition (3) is thus satisfied with "plenty to spare."

To prove the weaker version, consider the class Φ of functions ϕ which have a continuous second derivative and which vanish outside a bounded interval. Let $\phi(\xi) \in \Phi$, and set

$$F(x) = \int_{-\infty}^{\infty} \phi(\xi)e^{ix\xi}\, d\xi. \tag{4}$$

Clearly, $F(x) \in L(-\infty, \infty)$, and $|F(x)|\ |K(x-y)|\ |m(y)|$ is integrable as a function of (x, y), where K and m obey the hypotheses above. Hence, using (2) and Fubini's theorem, we have

$$\begin{aligned} 0 &= \int_{-\infty}^{\infty} F(x)\left(\int_{-\infty}^{\infty} K(x-y)m(y)\, dy\right) dx \\ &= \int_{-\infty}^{\infty} m(y)\left(\int_{-\infty}^{\infty} K(x-y)F(x)\, dx\right) dy \end{aligned} \tag{5}$$

and clearly

$$\int_{-\infty}^{\infty} K(x-y)F(x)\, dx = \int_{-\infty}^{\infty} \kappa(\xi)\phi(\xi)e^{i\xi y}\, d\xi. \tag{6}$$

Thus, for each function ϕ in Φ, we will have

$$0 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} m(y)\kappa(\xi)\phi(\xi)e^{i\xi y}\, d\xi\, dy. \tag{7}$$

The stronger requirement (3) on $K(x)$ implies that its transform $\kappa(\xi)$ has a continuous second derivative; since, by assumption (1), $\kappa(\xi)$ is never zero, we see that multiplication by κ carries the class Φ into itself exactly: $\kappa\Phi = \Phi$. We can rewrite (7) as

$$0 = \int_{-\infty}^{\infty} m(y)\left(\int_{-\infty}^{\infty} \phi(\xi)e^{i\xi y}\, d\xi\right) dy \tag{8}$$

for every function ϕ in the class Φ. Since Φ is closed under translation, we can replace $\phi(\xi)$ by $\phi(\xi-\alpha)$ and apply the usual change of variable to arrive at

$$0 = \int_{-\infty}^{\infty} m(y) \left(\int_{-\infty}^{\infty} \phi(\xi) e^{i\xi y} \, d\xi \right) e^{i\alpha y} \, dy \tag{9}$$

holding now for all real α. Using (4), this may be written as

$$0 = \int_{-\infty}^{\infty} m(y) F(y) e^{i\alpha y} \, dy \tag{10}$$

for all real α. By the uniqueness of Fourier transforms, we may conclude that

$$m(y)F(y) = 0 \tag{11}$$

for almost all y.

Since ϕ has compact support, $F(y)$ is an entire function, and can be chosen not to be identically zero. Since it can then have at most a denumerable number of zeros, $m(y) = 0$ for almost all y, and the proof is complete.

It should perhaps be pointed out that the proof above uses implicitly the concept of a generalized Fourier integral (forced upon us by the fact that $m(y)$ is merely bounded). Also, the relation $\kappa\Phi = \Phi$ is somewhat reminiscent of the algebraic nature of the Tauberian theorem.

Reference

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