

ORTHOGONAL EXPANSIONS WITH POSITIVE COEFFICIENTS

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In this note we give a simple proof of a generalization of a theorem of Szegő. The theorem in question is

$$(1) \quad (\sin \theta)^{2\lambda-1} P_n^{(\lambda)}(\cos \theta) = \sum_{k=0}^{\infty} \alpha_{k,n}^{\lambda} \sin(n+2k+1)\theta$$

where $\lambda > 0$, $\lambda \neq 1, 2, \dots$, and

$$\alpha_{k,n}^{\lambda} = \frac{2^{2-2\lambda}(n+k)! \Gamma(n+2\lambda) \Gamma(k+1-\lambda)}{\Gamma(\lambda) \Gamma(1-\lambda) k! n! \Gamma(n+k+\lambda+1)}$$

and $P_n^{(\lambda)}(\cos \theta)$ is given by

$$(1 - 2r \cos \theta + r^2)^{-\lambda} = \sum_{n=0}^{\infty} P_n^{(\lambda)}(\cos \theta) r^n.$$

Recall that $\sin(n+1)\theta/\sin \theta = P_n^{(1)}(\cos \theta)$ so that (1) may be written

$$(\sin \theta)^{2\lambda-1} P_n^{(\lambda)}(\cos \theta) = \sum_{k=0}^{\infty} \alpha_{k,n}^{\lambda} P_{n+2k}^{(1)}(\cos \theta) (\sin \theta)$$

or

$$(\sin \theta)^{2\lambda} P_n^{(\lambda)}(\cos \theta) = \sum_{k=0}^{\infty} \alpha_{k,n}^{\lambda} P_{n+2k}^{(1)}(\cos \theta) (\sin \theta)^2.$$

This suggests that a formula of the form

$$(2) \quad (\sin \theta)^{2\lambda} P_n^{(\lambda)}(\cos \theta) = \sum_{k=0}^{\infty} \alpha_{k,n}^{\lambda, \mu} P_n^{(\mu)}(\cos \theta) (\sin \theta)^{2\mu}$$

is true. Since $\alpha_{k,n}^{\lambda}$ is positive for $0 < \lambda < 1$ we might conjecture that $\alpha_{k,n}^{\lambda, \mu}$ is positive for $\lambda < \mu$. In fact, we show that it is for $(\mu-1)/2 < \lambda < \mu$. The condition $(\mu-1)/2 < \lambda$ is necessary to obtain convergence of the series (2).

This result follows from an old result of Gegenbauer [1] which has almost been forgotten by the mathematical community. Since $P_n^{(\mu)}(x)$ is a polynomial we may write

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$$(3) \quad P_n^{(\mu)}(x) = \sum_{k=0}^n \beta_{k,n}^{\lambda,\mu} P_k^{(\lambda)}(x).$$

Gegenbauer gives the value of β as

$$\beta_{k,n}^{\lambda,\mu} = \frac{\Gamma(\lambda)(k+\lambda)\Gamma((n-k)/2+\mu-\lambda)\Gamma(\mu+(k+n)/2)}{\Gamma(\mu)[(n-k)/2]!\Gamma(\mu-\lambda)\Gamma(\lambda+1+(k+n)/2)}$$

if $n-k$ is even, $n \geq k$, and $\beta_{k,n}^{\lambda,\mu} = 0$ otherwise. Observe that $\beta \geq 0$ if $\mu \geq \lambda$. A simple proof of this is in [2].

We also need the following simple fact. Let $w(x)$ and $w_1(x)$ be positive functions on a set E . Let $\{p_n(x)\}$ and $\{q_n(x)\}$ be the orthonormal polynomials associated with $w(x)$ and $w_1(x)$ respectively. Then if

$$(4) \quad q_n(x) = \sum_{k=0}^n c_{k,n} p_k(x)$$

we have

$$(5) \quad w(x)p_k(x) = \sum_{n=k}^{\infty} c_{k,n} q_n(x)w_1(x).$$

The convergence of the series can be taken in the appropriate L^2 space if $[w(x)]^2/w_1(x)$ is integrable. (5) follows immediately from (4) since the Fourier coefficients are the same in the two expansions.

Using (3), (4) and (5) and remembering that $\{P_n^{(\lambda)}(x)\}$ are orthogonal but not orthonormal we get

$$(6) \quad (1-x^2)^{\lambda-1/2} P_n^{(\lambda)}(x) = \sum_{k=0}^{\infty} \alpha_{k,n}^{\lambda,\mu} P_{n+2k}^{(\mu)}(x) (1-x^2)^{\mu-1/2}$$

where

$$\alpha_{k,n}^{\lambda,\mu} = \frac{\Gamma(\mu)2^{2\mu-2\lambda}(n+2k+\mu)(n+2k)!\Gamma(n+2\lambda)\Gamma(n+k+\mu)\Gamma(k+\mu-\lambda)}{\Gamma(\mu-\lambda)\Gamma(\lambda)n!k!\Gamma(n+k+\lambda+1)\Gamma(n+2k+2\mu)}.$$

Using the asymptotic formula for $P_n^{(\lambda)}(x)$ it is easy to see that the series in (6) converges for $-1 < x < 1$ if $\lambda > (\mu-1)/2$. Also observe that $\alpha_{k,n}^{\lambda,\mu} > 0$ for $(\mu-1)/2 < \lambda < \mu$.

Setting $n=0$ in (6) we get

$$(7) \quad (1-x^2)^{-\alpha} = \frac{\Gamma(\mu)2^{2\alpha}\Gamma(2\mu-2\alpha)}{\Gamma(\alpha)\Gamma(\mu-\alpha)} \cdot \sum_{k=0}^{\infty} \frac{(2k+\mu)(2k)!\Gamma(k+\mu)\Gamma(k+\alpha)}{k!\Gamma(k+\mu-\alpha+1)\Gamma(2k+2\mu)} P_{2k}^{(\mu)}(x).$$

For $\alpha = 1/2$ and $\mu = 1/2$ this is due to Bauer and $\alpha = 1/2$, $\mu > 0$ it is due to Gegenbauer [1].

There are two other instances of polynomial expansions of different orthogonal polynomials that involve the classical polynomials and have positive coefficients.

The better known one is

$$(8) \quad L_n^{(\beta)}(x) = \sum_{k=0}^n \frac{\Gamma(\beta - \alpha + n - k)}{\Gamma(\beta - \alpha)(n - k)!} L_k^{(\alpha)}(x),$$

where $L_n^{(\beta)}(x)$ are the Laguerre polynomials of order β , degree n , and $\beta > \alpha$. Notice that the coefficients are positive. See [3, p. 209] for (8).

For Jacobi polynomials, $P_n^{(\alpha, \beta)}(x)$, the following expansion holds

$$P_n^{(\beta, \gamma)}(x) = \frac{\Gamma(n + \gamma + 1)}{\Gamma(\beta - \alpha)\Gamma(n + \gamma + \beta + 1)} \sum_{k=0}^n \frac{\Gamma(n + k + \beta + \gamma + 1)\Gamma(n - k + \beta - \alpha)\Gamma(k + \alpha + \gamma + 1)(2k + \alpha + \gamma + 1)}{\Gamma(n + k + \alpha + \gamma + 2)\Gamma(n - k + 1)\Gamma(k + \gamma + 1)} P_k^{(\alpha, \gamma)}(x).$$

See [4, p. 254]. Again notice that, if $\gamma > -1$ and $\beta > \alpha \geq 0$, the coefficients are positive. The common feature of both of these results is that the polynomials are normalized correctly at the right point. In the Laguerre case $L_n^{(\alpha)}(0) > 0$ and $P_n^{(\alpha, \beta)}(1) > 0$ for Jacobi polynomials. An interesting conjecture can be formulated of which these are a special case. For convenience we formulate it on $(0, \infty)$.

Let $w(x)$ be a positive function on $(0, \infty)$ such that $\int_0^\infty x^n w(x) dx$ exists for each $n = 0, 1, \dots$. Let $\{p_n(x)\}$ be the orthonormal polynomials with respect to $w(x)$ normalized by $p_n(0) > 0$. All of the zeros of $p_n(x)$ are in $(0, \infty)$ so this is possible. Let $\{p_n^\alpha(x)\}$ be the polynomials orthonormal with respect to $x^\alpha w(x)$ normalized in the same way. Then if $\alpha > 0$ we conjecture that

$$p_n^\alpha(x) = \sum_{k=0}^n \alpha_k p_k(x)$$

with $\alpha_k > 0$.

For α an integer this follows from known classical results. For if $\alpha = 1$ then

$$c_n p_n^1(x) = \frac{p_{n+1}(0)p_n(x) - p_n(0)p_{n+1}(x)}{x}$$

by a theorem of Christoffel [4, Theorem 2.5]. By another theorem of Christoffel [4, Theorem 3.2.2],

$$\frac{p_{n+1}(0)p_n(x) - p_n(0)p_{n+1}(x)}{x} = r_n \sum_{k=0}^n p_k(0)p_k(x).$$

Thus

$$p_n^1(x) = r_n \sum_{k=0}^n p_k(0)p_k(x)$$

and $p_k(0)$, $p_n^1(0)$, and thus r_n are positive by assumption. For larger integral values of α the result follows by iteration.

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