ECHELON SPACES OF ORDER ∞

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The notion of Echelon spaces was introduced by Köthe [2] as a means of constructing sequence spaces by taking the intersection of multiples of l^1 (see below for definitions). The definition was extended by Dieudonné and Gomes [1] to include l^p for $1 \leqq p < \infty$. It is the purpose of this note to consider the case, $p = \infty$. All of the terminology defined below is due to Köthe.

A sequence of complex numbers whose ith coordinate is x_i will be denoted by x or (x_i). A sequence of sequences such that the ith coordinate of the kth sequence is x_i^k will be denoted by $(x^{(k)})$. If x, y are sequences and α is a complex number, we denote the pointwise sum, pointwise product and pointwise scalar product by $x+y$, xy, αx respectively. If each y_i is different from zero, we define the pointwise quotient, x/y. If A is a set of sequences, then $xA = \{xy \mid y \in A\}$. We define an ordering which is compatible with these arithmetic operations by taking as positive cone, $P = \{x \mid x_i \geqq 0$ for all i and $x_i \neq 0$ for at least one $i\}$.

The symbol, λ, will denote a set of sequences which is a vector space under the above operations. Given λ, we define its *α-dual*, $\lambda^\times = \{u \mid x \in \lambda \Rightarrow \sum_{i=1}^\infty |x_i u_i| < \infty\}$. The relation, $\langle x, u\rangle = \sum_{i=1}^\infty x_i u_i$ places λ, $\lambda^\times$ in duality. For example, if $1 \leqq p \leqq \infty$ and we let $l^p = \{x \mid \sum_{i=1}^\infty |x_i|^p < \infty\}$ (if $p = \infty$, the sum is replaced by the sup), then if $\lambda = l^p$, $\lambda^\times = l^q$ where $1/p + 1/q = 1$.

The vector space, λ, is said to be *perfect* if $\lambda = \lambda^{\times\times}$. It is easy to see that $\lambda^\times$ is always perfect. If A is a set of sequences, we define the *normal hull* $A^n = \{x \mid \exists y \in A \ni |x_i| \leqq |y_i|$ for all $i\}$. If $A = A^n$, then A is said to be *normal*. Since λ, $\lambda^\times$ are in duality, we may speak of simply bounded subsets of λ. Now $\lambda^\times$ is clearly normal and from this it follows that the normal hull of a simply bounded set is simply bounded (as a subset of $\lambda^{\times\times}$ perhaps, if λ is not normal).

Let $(a^{(k)})$ be a sequence of sequences such that $a_i^k > 0$ for all i, k and $a^{(k)} \leqq a^{(k+1)}$. If $1 \leqq p \leqq \infty$, we define the *Echelon space of order p* corresponding to (a^k) to be,

$$\lambda = \bigcap_{k=1}^{\infty} \frac{1}{a^{(k)}} l^p.$$

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Köthe [3, §30, 8 and 9] presents a complete discussion (due to himself and Dieudonné and Gomes) of Echelon spaces of order $p<\infty$. The main results are: λ is perfect; $\lambda^{\times}=\bigcup a^{(k)}l^{q}$; a characterization of bounded sets in λ, $\lambda^{\times}$; a necessary and sufficient condition that λ is a Montel space; and λ is always reflexive if $p>1$, but reflexive if and only if λ is a Montel space, when $p=1$. In the sequel, we consider these results for the case, $p=\infty$. The proofs of Propositions 1, 2 are direct computations with sequences and are different from those given by Köthe. The proof of Theorem 1 is only a slight variation of Köthe's proof but is presented here because of its delicacy.

We shall say that (a^{k}) is *strongly increasing* if there is no triple $\{(j_{\nu}), k_0, (M_k)\}$ where (j_{ν}) is a monotone increasing sequence of positive integers, k_0 is an integer and (M_k) is a sequence of positive numbers such that

$$a_{j_{\nu}}^{(k)} \leqq M_k\, a_{j_{\nu}}^{(k_0)}$$

for all $k\geqq k_0$. In short, we require that there is no subsequence of indices on which all of the $a^{(k)}$ are "dominated" by one of them.

In all that follows, $(a^{(k)})$ is a sequence of coordinate-wise monotone increasing sequences of positive numbers and λ is the Echelon space of order ∞ corresponding to $a^{(k)}$.

PROPOSITION 1. *The α-dual, $\lambda^{\times}$, of λ is given by $\lambda^{\times}=\bigcup_{k=1}^{\infty} a^{(k)}l^{1}$. λ is perfect.*

PROOF. It is easy to see that the union is contained in $\lambda^{\times}$, since l^1 is the α-dual of l^{∞}. Suppose $u\in\lambda^{x}$. We may assume that $u>0$. Let $b^{(k)}=u/a^{(k)}$. If u is not in any $a^{(k)}l^1$, then for each k we have, $b^{(k)}>0$, $\sum_{i=1}^{\infty} b_i^k=\infty$, and $b^{(k+1)}\leqq b^{(k)}$. Hence we can find a sequence $v=(v_i)$ and a strictly increasing sequence of positive integers, (i_k) such that $v_i=b_i^k$ for $i_k\leqq i<i_{k+1}$ and $\sum_{i=i_k}^{i_{k+1}-1} v_i>1$. Since $(b^{(k)})$ is decreasing, $v_i\leqq b_i^k$ for $i\geqq i_k$. Further, $v_i=0$ if $u_i=0$. Hence there exist positive numbers M_1, M_2, $\cdots$ such that $v_i\leqq M_k b_i^k$ for all i. Let x_i be 0 if $u_i=0$ and let $x_i=v_i/u_i$ otherwise. The sequence $x=(x_i)\in\lambda$ but $\sum x_i u_i$ diverges which is a contradiction. Hence u is in the union.

Finally it follows from [3, §30, 4, (1), b)] and the fact that l^{∞} is perfect, that λ is perfect.

It is shown by Köthe [3, §30, 5, (5)] that in a perfect space, weak and strong bounded sets are the same, so we shall use the term, bounded.

PROPOSITION 2. *A subset, B, of λ is bounded if and only if, for each k, the set, $a^{(k)}B$ is a norm bounded subset of l^{∞}.*

A subset, B, of $\lambda^\times$ is bounded if and only if there exists k_0 such that $B \subset a^{(k_0)} l^1$ and $(1/a^{(k_0)})B$ is a norm bounded subset of l^1.

PROOF. Everything is straightforward except the necessity of the condition in the second statement.

We shall first show that if $(u^{(k)})$ is a sequence of elements of $\lambda^\times$ such that $u^{(k)} \notin a^{(k)} l^1$ then $(u^{(k)})$ is not bounded. If this hypothesis holds, we can find a strictly increasing sequence of integers, (i_k) such that

$$\sum_{i=i_k}^{i_{k+1}-1} \frac{|u_i^k|}{a_i^k} > k.$$

Let $x^{(k)}$ be a sequence defined by

$$x_i^k = \begin{cases} \dfrac{\arg (u_i^k)}{a_i^k} & \text{for } i_k \leqq i < i_{k+1}, \\ 0 & \text{otherwise.} \end{cases}$$

Since $(a^{(k)})$ is increasing, $|x_i^k| \leqq 1/a_i^{k'}$ for $i \geqq i_{k'}$, $k \geqq k'$. Hence $(x^{(k)})$ is a bounded subset of λ. But $|\langle x^{(k)}, u^{(k)} \rangle| > k$, so $(u^{(k)})$ is not bounded. Thus we have shown that if B is a bounded subset of $\lambda^\times$, there exists k_0 such that $B \subset a^{k_0} l^1$.

Finally, suppose $(u^{(k)})$ is a sequence in the bounded set B, but

$$\lim_{k \to \infty} \sum_{i=1}^{\infty} \frac{|u_i^k|}{a_i^k} = \infty.$$

We shall choose subsequences (i_ν), (k_ν) inductively according to conditions which will be stated inductively. Let $i_1 = k_1 = 1$, and suppose the choice has been made up to ν. Since $(u^{(k)})$ is a bounded subset of $\lambda^\times$, it is coordinatewise bounded, so with the fact that $(a^{(k)})$ is increasing, we may define

$$M_\nu = \sup \left\{ \frac{|u_i^k|}{a_i^k} \,\middle|\, 1 \leqq i < i_\nu,\ k = 1, 2, \cdots \right\}.$$

Then we choose $k_{\nu+1}$, $i_{\nu+1}$ to be greater than k_ν, i_ν and such that

$$\sum_{i=1}^{i_{\nu+1}-1} \frac{|u_i^{k_\nu}|}{a_i^{k_\nu}} > \nu + (i_\nu - 1) M_\nu.$$

Hence,

$$\sum_{i=i_\nu}^{i_{\nu+1}-1} \frac{|u_i^{k_\nu}|}{a_i^{k_\nu}} > \nu.$$

Having chosen these sequences, we define, for each ν, the sequence $x^{(\nu)}$ by

$$x_i^\nu = \begin{cases} \dfrac{\arg(u_i^{k_\nu})}{a_i^{k_\nu}} & \text{for } i_\nu \leqq i < i_{\nu+1}, \\ 0 & \text{otherwise.} \end{cases}$$

As before, $(x^{(\nu)})$ is a bounded subset of λ, but $|\langle x^{(\nu)}, u^{(k_\nu)}\rangle| > \nu$ which is a contradiction.

THEOREM 1. *The spaces* λ, $\lambda^\times$ *are Montel spaces in their Mackey topologies if and only if* $(a^{(k)})$ *is strongly increasing.*

PROOF. If λ, $\lambda^\times$ are Montel spaces, the proof that $(a^{(k)})$ is strongly increasing is exactly the same as that given by Köthe [**3**, §30, 9, (1)].

Suppose (a^k) is strongly increasing. To show that λ (and hence $\lambda^\times$) is a Montel space, we must according to Köthe [**3**, §30, 7, 8] show that in λ, weak convergence implies strong convergence and in $\lambda^\times$, sections are strongly convergent.

The latter actually does not require the assumption that $(a^{(k)})$ is strongly increasing. In fact, if $u \in \lambda^\times$, then there exists k_0 such that $v = u/a^{(k_0)} \in l^1$. If B is a bounded subset of λ, then by Proposition 2, $a^{(k_0)}B$ is a bounded subset of l^∞. Let M be its bound. Let u^N, v^N be the Nth sections of u, v. Then for all $x \in B$,

$$|\langle x, u^N - u\rangle| = |\langle a^{k_0}x, v^N - v\rangle| \leqq M \sum_{i=N}^{\infty} |v_i| \to 0 \text{ as } N \to \infty.$$

Now we suppose that in λ, weak convergence does not imply strong convergence and we shall show that $(a^{(k)})$ is not strongly increasing.

Let $(x^{(n)})$ be a sequence in λ which converges weakly but not strongly to 0. Hence there is a bounded sequence $(u^{(n)})$ in $\lambda^\times$ such that $|\langle x^{(n)}, u^{(n)}\rangle| \geqq 1$ for all n. Let B, $B^\times$ be the normal hull of the sets $\{x^{(n)}\}$, $\{u^{(n)}\}$ respectively. They are again bounded. By Proposition 2 there exists a sequence (m_k) of positive numbers, a positive number, A, and an integer, k_0, such that $a_i^k|x_i| \leqq m_k$ for all $x \in B$ and for all $i = 1, 2, \cdots$; and $\sum_{i=1}^\infty (1/a_i^{k_0})|u_i| \leqq A$ for all $u \in B^\times$. Choose M_k such that

$$\sum_{k=1}^{\infty} \frac{m_k}{M_k} \leqq \frac{1}{2A}.$$

We shall choose sequences (n_ν), (r_ν), (s_ν) of integers such that $r_1=1$, $s_{\nu-1}<r_\nu<s_\nu$ for $\nu>1$ and $\sum_{j=r_\nu}^{s_\nu}|x_j^{n_\nu}u_j^{n_\nu}|\geqq 1/2$. Clearly this can be done for $\nu=1$. Suppose it has been done for ν. Choose $n_{\nu+1}$ large enough so that $|x_i^{n_{\nu+1}}|<1/3Aa_i^{k_0}$ for $i=1,\cdots,s_\nu$. This can be done since $(x^{(n)})$ is weakly convergent and hence coordinatewise convergent to 0. Hence,

$$\sum_{i=1}^{s_\nu}|x_i^{n_{\nu+1}}u_i^{n_{\nu+1}}|<\frac{1}{3A}\sum_{i=1}^{s_\nu}\frac{1}{a_i^{k_0}}|u_i^{n_{\nu+1}}|<\frac{1}{3},$$

so

$$\sum_{i=s_\nu+1}^{\infty}|x_i^{n_{\nu+1}}u_i^{n_{\nu+1}}|\geqq\frac{2}{3}>\frac{1}{2}\cdot$$

Hence we may choose $r_{\nu+1}=s_\nu+1$ and $s_{\nu+1}$ large enough to satisfy the stated inequality.

Now, let $J_{\nu,k}=\{j\,|\,r_\nu\leqq j\leqq s_\nu \text{ and } a_j^k>a_j^{k_0}M_k\}$. Suppose, for a given ν, $\bigcup_k J_{\nu,k}$ is the set of all integers from r_ν to s_ν. Then,

$$\frac{1}{2}\leqq\sum_{j=r_\nu}^{s_\nu}|x_j^{n_\nu}u_j^{n_\nu}|\leqq\sum_{k=1}^{\infty}\sum_{j\in J_{\nu,k}}|x_j^{n_\nu}u_j^{n_\nu}|<\sum_{k=1}^{\infty}\frac{1}{M_k}\sum_{j\in J_{\nu,k}}\frac{a_j^k}{a_j^{k_0}}|x_j^{n_\nu}u_j^{n_\nu}|$$
$$\leqq\sum_{k=1}^{\infty}\frac{m_k}{M_k}\sum_{j=1}^{\infty}\frac{1}{a_j^{k_0}}|u_j^{n_\nu}|\leqq A\sum_{k=1}^{\infty}\frac{m_k}{M_k}\leqq\frac{1}{2},$$

and this is a contradiction. Hence, for all ν, there exists j_ν, such that $r_\nu\leqq j_\nu\leqq s_\nu$, but $a_{j_\nu}^k\leqq a_{j_\nu}^{k_0}M_k$ and this contradicts the fact that $(a^{(k)})$ is strongly increasing.

COROLLARY 1. *The spaces* λ, $\lambda^\times$ *are Montel spaces in their Mackey topologies if and only if* λ (*resp.* $\lambda^\times$) *has no stepspace which is a diagonal transform of* l^∞ (*resp.* l^1) (*see* [2, *p.* 411 *and p.* 413] *for definitions*).

PROOF. This follows immediately from the theorem.

COROLLARY 2. *The spaces* λ, $\lambda^\times$ *are reflexive in their Mackey topologies if and only if they are Montel spaces.*

PROOF. This follows immediately from Corollary 1 and the condition for reflexivity given by Köthe [3, §30, 7, (5)].

PROPOSITION 3. *The space* λ *is an* (F)-*space in its strong topology.*

PROOF. It follows from Proposition 2 that λ is metrisable. Since λ is perfect, it is complete in the normal topology [3, §30, 5, (7)]. Therefore λ is complete in its strong topology.

References

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ON EXTREME POINTS OF THE NUMERICAL RANGE OF NORMAL OPERATORS[1]

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Suppose A is a bounded normal operator on the Hilbert space H. Then the extreme points of the closure of the numerical range are in the spectrum of A. This follows because the convex hull of the spectrum is the closure of the numerical range and because the extreme points of the convex hull of a compact set are in the compact set. The object of this note is to point out that more can be said about the extreme points of the numerical range itself. Namely

THEOREM. *For normal operators the extreme points of the numerical range are in the point spectrum.*

PROOF. Let A be a normal operator on the Hilbert space H. Let $\Lambda(A)$, $W(A)$, $\Pi_0(A)$ denote the spectrum, numerical range, point spectrum of A respectively. Suppose that λ is an extreme point of $W(A)$. Then 0 is an extreme point of $W(A-\lambda)=W(A)-\lambda$. Also $\Pi_0(A-\lambda)=\Pi_0(A)-\lambda$. Thus for our purposes it is sufficient to show that if 0 is an extreme point of $W(A)$, then 0 is an eigenvalue.

Because $W(e^{i\theta}A)=e^{i\theta}W(A)$ and $\Pi_0(e^{i\theta}A)=e^{i\theta}\Pi_0(A)$, and since 0 is an extreme point of the convex set $W(A)$, we may assume that $W(A)$ lies entirely within the closed right-hand half plane Re $z\geqq 0$.

By the spectral theorem for normal operators, A is unitarily equivalent to a multiplication on $L_2(X, \mu)$ by a function $a(x)$ in $L_\infty(X, \mu)$ where (X, μ) is some finite measure space. That is, after a change of notation, $H=L_2(X, \mu)$ and $(Af)(x)=a(x)f(x)$ for all f in H.

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