ON THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF THE DIFFERENTIAL EQUATION $y''+p(x)y=0$

A. C. LAZER

G. Prodi [5] has shown that if $\lim_{x\to+\infty} p(x) = +\infty$ and if $p(x)$ is nondecreasing, then there exists at least one nontrivial solution of the differential equation

$$y'' + p(x)y = 0 \tag{L}$$

which tends to zero as x tends to infinity. In this note we will give another condition which guarantees this same property. Although Prodi's result follows from Theorem 2, below, when $p(x)$ is assumed to be absolutely continuous, our methods of proof will be entirely dissimilar from those used in [5]. For further literature on the asymptotic behavior of solutions of (L) under the hypothesis that $p(x)\to+\infty$ as $x\to+\infty$, the reader may consult [2, §5.5]. A more recent result is contained in [3].

All integrals appearing in this note are Lebesgue integrals.

THEOREM 1. *If $p(x)$ is positive and absolutely continuous on any finite subinterval of the half-axis $I: a \leqq x < +\infty$, and if for every solution $y(x)$ of (L), $\lim_{x\to+\infty} \int_a^x (y'/p)^2 p'dt$ exists and is finite, then there exists at least one nontrivial solution of (L) which tends to zero as x tends to infinity.*

PROOF. Since $p(x)$ is assumed to be absolutely continuous on any finite subinterval of I, $p'(x)$ exists almost everywhere on I. Moreover, for any solution $y(x)$ of (L), the function

$$G[y(x)] \equiv \frac{[y'(x)]^2}{p(x)} + (y(x))^2 \tag{1}$$

is absolutely continuous on any finite subinterval of I,

$$\frac{dG[y(x)]}{dx} = -p'(x)\left(\frac{y'(x)}{p(x)}\right)^2$$

almost everywhere, and for $x > a$,

$$G[y(x)] = G[y(a)] - \int_a^x \left(\frac{y'}{p}\right)^2 p'dt. \tag{2}$$

(See [4, p. 255].) By the conditions of the theorem $\lim_{x\to+\infty} G[y(x)]$

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exists and is finite, and since $0 \leqq (y(x))^2 \leqq G[y(x)]$, we infer immediately that all solutions of (L) must be bounded on I. Let $U_1(x)$ and $U_2(x)$ be two linearly independent solutions of (L) which satisfy the condition

$$U_1(x)U_2'(x) - U_2(x)U_1'(x) \equiv 1. \tag{3}$$

We may suppose that $U_1(x)$ does not tend to zero as x tends to infinity. Since $p(x) \to +\infty$ all solutions of (L) are oscillatory, in other words, vanish for arbitrarily large values of x. If $x_1 < x_2 < x_3 \cdots$ be the successive relative maximum points of the solution $U_1(x)$, then

$$U_1'(x_n) = 0, \qquad G[U_1(x_n)] = (U_1(x_n))^2 \tag{4}$$

and

$$\lim_{n \to +\infty} x_n = +\infty. \tag{5}$$

Therefore, since $\lim_{x \to +\infty} G[U_1(x)]$ exists, and $U_1(x_n) > 0$, $\lim_{n \to +\infty} U_1(x_n)$ exists and by the above assumption is equal to a positive number c. Let N be so large that $U_1(x_n) > c/2$, for $n \geqq N$. From (3) and (4) it follows that

$$|U_2'(x_n)| \leqq 2/c, \qquad n \geqq N. \tag{6}$$

Since $U_2(x)$ is bounded on I, there exists a sequence of integers $\{n_j\}$ such that the sequence $\{U_2(x_{n_j})\}$ converges to a number b. We consider the nontrivial solution

$$Z(x) = U_2(x) - (b/c)U_1(x).$$

From the above we see that

$$\lim_{n_j \to +\infty} Z(x_{n_j}) = 0$$

and from (4) and (6)

$$|Z'(x_{n_j})| = |U_2'(x_{n_j})| \leqq 2/c.$$

Hence,

$$0 \leqq G[Z(x_{n_j})] \leqq \frac{4}{c^2 p(x_{n_j})} + (Z(x_{n_j}))^2,$$

for $n_j \geqq N$, and since $\lim_{x \to +\infty} p(x) = +\infty$, it follows by (5) that $\lim_{n_j \to +\infty} G[Z(x_{n_j})] = 0$. As was shown above, $\lim_{x \to +\infty} G[Z(x)]$ exists so that $\lim_{x \to +\infty} G[Z(x)] = 0$. Hence, from the inequality $0 \leqq (Z(x))^2 \leqq G[Z(x)]$, we see at once that $\lim_{x \to +\infty} Z(x) = 0$. This completes the proof of Theorem 1.

THEOREM 2. *If $p(x)$ is positive, absolutely continuous on every finite subinterval of the half-axis $I: a \leqq x < +\infty$, $\lim_{x\to+\infty} p(x) = +\infty$, and*

$$\lim_{x\to+\infty} \int_a^x \frac{(|p'| - p')}{p} dt$$

is finite, then there exists at least one nontrivial solution of (L) which tends to zero as x tends to infinity.

PROOF. We introduce the following notation:

$$(p'(x))^+ \equiv \frac{|p'(x)| + p'(x)}{2},$$

$$(p'(x))^- = \frac{|p'(x)| - p'(x)}{2}.$$

To prove Theorem 2, it is sufficient, by Theorem 1, to show that

$$\lim_{x\to+\infty} \int_a^x \left(\frac{y'}{p}\right)^2 p' dt$$

exists and is finite for every solution $y(x)$ of (L). If $y(x)$ is any solution of (L), then by (1) and (2)

$$0 \leqq \frac{[y'(x)]^2}{p(x)} \leqq \frac{[y'(x)]^2}{p(x)} + (y(x))^2 \equiv G[y(x)]$$
$$= G[y(a)] - \int_a^x \left(\frac{y'}{p}\right)^2 (p')^+ dt + \int_a^x \left(\frac{y'}{p}\right)^2 (p')^- dt,$$

so that

$$\frac{[y'(x)]^2}{p(x)} \leqq G[y(a)] + \int_a^x \left(\frac{y'}{p}\right)^2 (p')^- dt; \tag{7}$$

$$\int_a^x \left(\frac{y'}{p}\right)^2 (p')^+ dt \leqq G[y(a)] + \int_a^x \left(\frac{y'}{p}\right)^2 (p')^- dt. \tag{8}$$

By application of Bellman's lemma [1, p. 35] to the inequality (7), we infer that

$$\frac{[y'(x)]^2}{p(x)} \leqq G[y(a)] \exp\left(\int_a^x \frac{(p')^-}{p} dt\right)$$
$$\leqq G[y(a)] \exp\left(\int_a^\infty \frac{(p')^-}{p} dt\right).$$

By hypothesis $\int_a^\infty ((p')^-/p)\, dt$ is finite, and hence $[y'(x)/p(x)]^2$ is bounded on $[a, \infty]$. Thus, $\lim_{x\to+\infty} \int_a^x (y'/p)^2 (p')^- \, dt$ exists and is finite, and by (8) the same statement holds for

$$\lim_{x\to+\infty} \int_a^x \left(\frac{y'}{p}\right)^2 (p')^+ \, dt.$$

Hence

$$\lim_{x\to+\infty} \int_a^x \left(\frac{y'}{p}\right)^2 p' dt = \lim_{x\to+\infty} \left(\int_a^x \left(\frac{y'}{p}\right)^2 (p')^+ \, dt - \int_a^x \left(\frac{y'}{p}\right)^2 (p')^- \, dt \right)$$

exists and is finite. The assertion in Theorem 2 now follows from Theorem 1.

References

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Western Reserve University