HILBERT MANIFOLDS WITHOUT EPICONJUGATE POINTS

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1. Introduction. In 1961, S. Kobayashi [**4**] proved the following:

THEOREM. *Let M be a complete Riemannian manifold. If there exists a point p of M such that no geodesic passing through p contains a point conjugate to p, then the universal covering space of M is diffeomorphic to Euclidean space. More precisely, the exponential map* $\exp_p$: $M_p \to M$ *is a covering map.*

This theorem had been proved by Myers [**6**] under the assumption that M was analytic, and this assumption was essential in his proof. Kobayashi, and also Helgason in his book [**3**], showed that analyticity was superfluous and could be replaced with mere smoothness of a sufficiently high order.

We want to consider this theorem for infinite-dimensional Riemannian Hilbert manifolds. We do not know if it is true in unrestricted generality; the introduction in §3 of an *ad hoc* assumption on the metric is necessary for our method to work. But the class of manifolds satisfying the assumption is large, including all negatively curved manifolds as well as some which are everywhere positively curved. The main results are stated in Theorems 4.1 and 4.4.

Our definition of completeness is as follows: a manifold M is complete if it is Cauchy complete in its intrinsic topological metric. Unfortunately, infinite-dimensional manifolds complete in this sense may be incomplete in another sense: they may bear two points unconnectable by a minimal geodesic (see Example 5.1).

The question of conjugate points arises. They may be defined in the finite-dimensional case as singularities of the exponential map. We define them similarly in the infinite-dimensional case but find in §2 that two types appear, which we call monoconjugate and epiconjugate. The epiconjugate are the more sensitive measure of pathology. §5 contains examples demonstrating that pathological distributions of conjugate points can occur. One effect of the *ad hoc* assumption of §3 is to rule out the possibility of conjugate points.

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paper was in a preliminary stage. The results of this paper, obtained under stronger hypotheses, form part of the author's dissertation [2]. The author wishes to express his gratitude to his teacher, Professor Leon W. Green, for his advice and encouragement.

2. **Conjugate points.** A recent book of Lang [5] has exposed the foundations of Riemannian geometry on infinite-dimensional Hilbert manifolds, showing that many of the objects of classical local Riemannian geometry exist regardless of whether or not the dimension is finite. We assume that M is a smooth connected Hilbert manifold supplied with a Riemannian metric g. To save repetition and needless exceptions, M will be infinite dimensional unless otherwise stated. Then, exactly as in the Euclidean case, M bears a unique symmetric connection ∇ compatible with g and the curvature mapping R is given in terms of the Lie bracket by $R(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$. We may speak of covariant differentiation along curves, of geodesics, and of the exponential maps Exp and exp. There exist at each point of M normal coordinate neighborhoods and Whitehead simple convex neighborhoods.

Let $L(\gamma)$ be the length of the smooth arc γ. The *intrinsic metric* ρ is defined on M by $\rho(x, y) = \inf\{L(\gamma) \mid \gamma$ is a smooth arc from x to $y\}$. Define M to be *complete* if and only if it is complete with respect to ρ and assume from this point on that *all manifolds considered are complete.* Exactly as in the Euclidean case, we can prove that geodesics can be continued for all values of the arc length so that, for each p in M, the domain of $\exp_p$ is all of the tangent space M_p.

Fix p in M and v in M_p. Let γ be a geodesic issuing from p in the same direction as v, parametrized by arc length, with $\gamma(0) = p$. Let τ_s: $M_p \to M_{\gamma(s)}$ be parallel translation along γ. We are interested in studying the singularities of $\exp_p = \exp$, so want to compute $d\exp_v$. It is well known that $d\exp_v$ can be expressed in terms of the solution of a certain linear differential equation, the Jacobi differential equation (JDE). If J: $\boldsymbol{R} \to M_p$, then JDE is

$$J''(s) + \tau_s^{-1}R(\gamma'(s), \tau_s J(s))\gamma'(s) = 0.$$

It is convenient to define a family K of linear transformations of M_p into itself by $K(s)y = \tau_s^{-1}R(\gamma'(s), \tau_s y)\gamma'(s)$, so that the JDE becomes

$$J''(s) + K(s)J(s) = 0. \tag{2.1}$$

Make the standard identification of $(M_p)_v$ with $M_p \times M_p$ so that w in $(M_p)_v$ can be considered as lying in M_p itself. Let B_w be a solution of (2.1) with $B_w(0) = 0$ and $B_w'(0) = w$. Then, if $\sigma = \|v\|$, $d\exp_v w = \tau_\sigma(\sigma^{-1}B_w(\sigma))$. It is easy to see that $d\exp_v v = \tau_\sigma v$.

Let $q=\exp v$. We say that q is *epiconjugate* (resp. *monoconjugate*) to p along γ if $d\exp_v$ is not epimorphic (resp. monomorphic). It will follow from later calculations that each species is symmetric in p and q. Since τ_s is an isometry and scalar multiplication is an isomorphism, it is enough to look at the solutions B of the Jacobi equation (2.1) to determine conjugacy. In terms of the solution B_w introduced above, s gives an epiconjugate point if there can be found a y in M_p with no w in M_p for which $B_w(s)=y$. Similarly, s gives a monoconjugate point if $B_w(s)=0$ for some w.

If M is finite dimensional, these two species coincide, but they are not the same on infinite-dimensional manifolds in general, as we will show later by example. We can prove:

THEOREM 2.1. *q monoconjugate to p implies q epiconjugate to p. q epiconjugate to p and image $d\exp_v$ closed imply q monoconjugate to p.*

To prove Theorem 2.1, we will use the adjoint to $d\exp_v$. For convenience, let $\Delta_v=\tau_\sigma^{-1}d\exp_v$ and continue to let $\sigma=\|v\|$. Let $E(s)=d^2/ds^2+K(s)$. One of the symmetries of R is equivalent to $K(s)=K(s)^*$ so that the operator $E(s)$ is formally self-adjoint. Letting $\langle\cdot,\cdot\rangle$ be the inner product in M_p, we apply the Green's integral identity on the interval $[0,\sigma]$ to vector fields A and B, getting

$$
\begin{aligned}
(2.2)\quad &\int_0^\sigma \{\langle E(s)A(s), B(s)\rangle - \langle A(s), E(s)B(s)\rangle\}ds \\
&= \langle A'(\sigma), B(\sigma)\rangle - \langle A(\sigma), B'(\sigma)\rangle - \langle A'(0), B(0)\rangle + \langle A(0), B'(0)\rangle.
\end{aligned}
$$

Suppose that A satisfies $E(s)A(s)\equiv 0$, $A(0)=0$, and $A'(0)=w$, while B satisfies $E(s)B(s)\equiv 0$ and $B(\sigma)=0$. After dividing (2.2) by σ, we find

$$(2.3)\qquad \langle \Delta_v w, B'(\sigma)\rangle = \langle w, -\sigma^{-1}B(0)\rangle.$$

From (2.3), we derive a method for calculating Δ_v^*. Let $k(s)=K(\sigma-s)$ and B satisfy

$$(2.4)\qquad B''(s)+k(s)B(s)=0$$

on $[0,\sigma]$ subject to $B(0)=0$ and $B'(0)=y$. Then $\Delta_v^*y=\sigma^{-1}B(\sigma)$. In general terms, Δ_v^* is calculated by solving the JDE backwards along γ from σ to 0.

If C is a solution of (2.1) on $[0,\sigma]$, then c given by $c(s)=C(\sigma-s)$ is a solution of (2.4) on $[0,\sigma]$. If $C(0)=C(\sigma)=0$, then $c(0)=c(\sigma)=0$. Thus, the correspondence $C\leftrightarrow c$ sets up a one-to-one correspondence of kernel Δ_v and kernel Δ_v^*. By general theorems on operators and adjoints, kernel Δ_v^* coincides with the orthogonal complement of the closure of image Δ_v. Therefore, if kernel Δ_v is nontrivial ($q=\exp v$ is monoconjugate), then kernel Δ_v^* is nontrivial, so image Δ_v can not be all of M_p ($q=\exp v$ is epiconjugate). On the other hand, if image Δ_v is closed, the argument reverses. Therefore we have proved Theorem 2.1.

We can now show the symmetry of conjugate pairs. Let us first remark that if $q=\exp v$ is on γ then $\exp_q$ has a differential along γ at p calculated by means of the appropriate JDE, which is (2.4). Therefore, if $p=\exp_q u$, $d\exp_{q,u}$ is determined (up to parallel translations) by Δ_v^*. It is easy to see that the pair (Δ_v, Δ_v^*) can only occur in the states $(\mathrm{I}_1, \mathrm{I}_1)$, $(\mathrm{II}_2, \mathrm{II}_2)$, and $(\mathrm{III}_3, \mathrm{III}_3)$ of the classification of Taylor [7], from which the symmetry is clear.

The two species of conjugate points introduced here have geometric significance. As in the finite-dimensional case, monoconjugate points are associated with nonminimality of geodesics. Epiconjugate points are associated with covering properties of the exponential map. If $q=\exp_p v$ is epiconjugate to p along γ, no neighborhood of q in M is fully covered by geodesics which issue from p and neighbor γ.

3. **An assumption.** Let p be a fixed point in M. M will be said to satisfy *condition* (δ) at p if:

for each $r>0$ there is a constant $\delta_r>0$ such that for every v in M_p with $\|v\|<r$ and for every w in $(M_p)_v$ there holds $\|d\exp_v w\| \geqq \delta_r \|w\|$.

If M has nonpositive sectional curvatures, it is classical [1, p. 342] that condition (δ) holds at every p in M with $\delta_r \equiv 1$. But a manifold may have everywhere positive sectional curvatures and still satisfy condition (δ) at some point. For example, a paraboloid of revolution in Euclidean three-space satisfies condition (δ) with its vertex as fixed point.

If γ is a geodesic issuing from p in the same direction as v, no point on γ is conjugate to p in either species. For at $q=\exp v$, if $d\exp_v w=0$ then $w=0$ by condition (δ). Also by condition (δ), image $d\exp_v$ is closed. Therefore, using Theorem 2.1, q is neither monoconjugate nor epiconjugate to p.

Conversely, suppose $q=\exp v$ not epiconjugate to p along γ. Then, by Theorem 2.1, q is not monoconjugate to p, so $d\exp_v$ is one-to-one, onto, and continuous. By a standard result of the theory of normed spaces [7, p. 180], $d\exp_v$ is bicontinuous. This means that for each v there is a $\delta_v>0$ such that $\|d\exp_v w\| \geqq \delta_v \|w\|$. But as $\|v\| \to \infty$, it may be that δ_v can approach 0 in such a way as to be unbounded away from 0 on any disc of large enough radius, so that condition (δ) may not be satisfied. Thus, condition (δ) appears stronger than requiring no conjugate points.

4. An infinite-dimensional form of the theorem of Myers and Kobayashi. We will prove:

THEOREM 4.1. *Let M be a complete simply-connected Hilbert manifold satisfying condition* (δ) *at p. Then M is homeomorphic to its local Hilbert space model.*

Let $T=\exp_p M_p$. We will show, first, that $T=M$ and, second, that $\exp_p$ is a covering map.

LEMMA 4.2. *T is open in M.*

PROOF. Since condition (δ) is satisfied at p, Δ_v has no kernel or cokernel. Therefore, it is a topological isomorphism. By the Inverse Function Theorem, $\exp_p$ is a local diffeomorphism and so T is open in M.

By Lemma 4.2, T is a submanifold of M and inherits a Riemannian structure.

LEMMA 4.3. *T is closed in M.*

PROOF. (See [1, p. 346].) Let q be a limit point of T and let λ: $[0, l]\to M$ be a smooth curve of length l, parametrized by arc length, with $\lambda(0)=p$ and $\lambda(l)=q$. We do not know yet that q can be reached from p along a geodesic but, since p has a normal neighborhood, there is an interval $[0, r)$ such that $\lambda([0, r))\subset T$. Let t be the supremum of r with this property. We will show that $\lambda(t)=q$ and that q is in T.

Since $\exp_p$ is a local diffeomorphism, λ can be lifted to a unique curve θ: $[0, t)\to M_p$ such that $\theta(0)=0$ and $\exp_p\cdot\theta=\lambda$. Applying condition ($\delta$) on the sphere $\{\|v\|<2l\}$ we find the inequality

$$L(\theta\,|\,[a, b]) \leqq \delta_{2l}^{-1}(b-a) \tag{4.1}$$

whenever $0\leqq a\leqq b\leqq t$. Let $s_1<s_2<\cdots$ be a sequence of reals with $\sup s_n=t$. The points $\lambda_n=\lambda(s_n)$ form a Cauchy sequence in M. By (4.1) the points $\theta_n=\theta(s_n)$ form a Cauchy sequence in M_p. M_p is complete; therefore $\theta_n\to y$ in M_p. By continuity, $\lambda(t)=\exp_p y$ and $\lambda(t)$ is an interior point of T since $\exp_p$ is a local diffeomorphism. Therefore $t=l$, $\lambda(t)=q$, and q is in T. This completes the proof of Lemma 4.3.

Lemmas 4.2 and 4.3 have made no appeal to simple connectedness and together they prove the first assertion of the following:

THEOREM 4.4. *Let M be a complete Hilbert manifold satisfying condition* (δ) *at p. Then* $\exp_p$ *is onto M. The pair* (M_p, $\exp_p$) *is a covering manifold of M.*

To complete the proof of Theorem 4.4, we may proceed exactly as in Kobayashi [**4**, §3], since only local properties of geodesics are used in his proof.

To complete the proof of Theorem 4.1, we note that, if M is simply connected, the uniqueness of the universal covering manifold implies that M_p and M are homeomorphic by way of $\exp_p$. Since $\exp_p$ is a local diffeomorphism, M and M_p are even globally diffeomorphic.

REMARK. We have shown that each point of M can be joined to p by a geodesic which is unique if M is simply connected. In this case, the unique geodesic is also minimal, since $\exp_p$ covers all of M by a single normal coordinate system.

5. **Pathological examples.** Let l_2 be the Hilbert space of real sequences $x=(x_i \mid i=1, 2, \cdots)$ such that $\|x\|=\sqrt{\sum x_i^2}<\infty$. Let $(a_i \mid i=1, 2, \cdots)$ be a sequence of positive real numbers bounded away from 0 and ∞. Then the set $M=\{x\in l_2 \mid \sum a_i x_i^2=1\}$ is a smooth connected Hilbert submanifold, complete in the induced Riemannian metric. This manifold is an ellipsoid, and is diffeomorphic with the unit sphere in l_2.

EXAMPLE 5.1 (A MISSING MINIMAL GEODESIC). Choose $a_i=(2-1/i)^2$. Since $a_1=1$, the points $N=(1, 0, 0, \cdots)$ and $S=(-1, 0, 0, \cdots)$ are on M. Even though it is clear that N and S together lie on infinitely many geodesic arcs, we are going to show that they are connected by no minimal geodesic.

Define $T: M\to M$ by $Tx=y$, where

$$y_1 = x_1, \; y_2 = 0, \; y_i = \frac{2-\dfrac{1}{i-1}}{2-\dfrac{1}{i}}\, x_{i-1} \quad \text{for } i \geqq 3.$$

Then T is a smooth map with only N and S as fixed points. It is easily seen that any smooth curve from N to S is taken by T into another such curve which is strictly shorter than the original. Therefore, there can be no minimal geodesic from N to S.

M is simply connected (even contractible over itself to a point). But all the sectional curvatures of M are >0, and condition (δ) is not satisfied since N has (infinitely many) conjugates along geodesics. This example should be compared with Theorem 4.1.

EXAMPLE 5.2 (PATHOLOGICAL DISTRIBUTION OF CONJUGATE POINTS). Let $a_1=a_2=1$, with N and S as in the last example. Let γ be the geodesic issuing from N given by $x_1 = \cos s$, $x_2 = \sin s$, $x_3 = x_4 = \cdots = 0$, where s represents arc length. N is given by $s = 0$, S by $s = \pi$.

One may now compute that the monoconjugate points to N on γ arise at the arc lengths $s = k\pi/\sqrt{a_i}$ for all integers k and for $i \geqq 3$. Suppose that $a_4 < a_5 < \cdots < \text{lub } a_i = 1$. Then $s = \pi$ is a limit point of a sequence of monoconjugate points given by $s = \pi/\sqrt{a_i}$. The limit point will be monoconjugate or not according as $a_3 = 1$ or $\neq 1$. We recall that in the finite-dimensional case, conjugate points may never have clustering arc lengths. If the limit point is not monoconjugate it is epiconjugate and in a pathological way, since one may show that the image of d exp is there a proper dense subset. The calculations are omitted since they are straightforward.

One may modify this procedure to produce a whole interval of conjugate points in a neighborhood of S, consisting of a dense subset of monoconjugate points with the remainder epiconjugate of the pathological type just discussed.

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