References

1. J. Dieudonné and A. P. Gomes, *Surs certains espaces vectoriels topologiques*, C. R. Acad. Sci. Paris **230** (1950), 1129–1130.
2. G. Köthe, *Die Stufenräume, eine einfache Klasse linearer vollkommener Räume*, Math. Z. **51** (1948), 317–345.
3. ———, *Topologische lineare Räume*. I, Springer, Berlin, 1960.

TULANE UNIVERSITY OF LOUISIANA

ON EXTREME POINTS OF THE NUMERICAL RANGE OF NORMAL OPERATORS[1]

C. R. MACCLUER

Suppose A is a bounded normal operator on the Hilbert space H. Then the extreme points of the closure of the numerical range are in the spectrum of A. This follows because the convex hull of the spectrum is the closure of the numerical range and because the extreme points of the convex hull of a compact set are in the compact set. The object of this note is to point out that more can be said about the extreme points of the numerical range itself. Namely

THEOREM. *For normal operators the extreme points of the numerical range are in the point spectrum.*

PROOF. Let A be a normal operator on the Hilbert space H. Let $\Lambda(A)$, $W(A)$, $\Pi_0(A)$ denote the spectrum, numerical range, point spectrum of A respectively. Suppose that λ is an extreme point of $W(A)$. Then 0 is an extreme point of $W(A-\lambda)=W(A)-\lambda$. Also $\Pi_0(A-\lambda)=\Pi_0(A)-\lambda$. Thus for our purposes it is sufficient to show that if 0 is an extreme point of $W(A)$, then 0 is an eigenvalue.

Because $W(e^{i\theta}A)=e^{i\theta}W(A)$ and $\Pi_0(e^{i\theta}A)=e^{i\theta}\Pi_0(A)$, and since 0 is an extreme point of the convex set $W(A)$, we may assume that $W(A)$ lies entirely within the closed right-hand half plane Re $z\geqq 0$.

By the spectral theorem for normal operators, A is unitarily equivalent to a multiplication on $L_2(X, \mu)$ by a function $a(x)$ in $L_\infty(X, \mu)$ where (X, μ) is some finite measure space. That is, after a change of notation, $H=L_2(X, \mu)$ and $(Af)(x)=a(x)f(x)$ for all f in H.

Received by the editors November 16, 1964.

[1] Research was sponsored by the National Science Foundation.

Because we have assumed that the numerical range of A is contained in the closed right-hand half plane and since the closure of the numerical range includes the spectrum, Re $a(x) \geqq 0$ a.e. Suppose 0 is not an eigenvalue. Then $|a(x)| > 0$ a.e. Let $E_1 = \{x \text{ in } X; \operatorname{Im} a(x) \geqq 0\}$ and $E_2 = \{x \text{ in } X; \operatorname{Im} a(x) < 0\}$. Then because 0 is an extreme point of $W(A)$ there is an f in H with $\|f\| = 1$ such that

$$\begin{aligned}(Af,f) = 0 &= \int_X a(x)\,|f(x)|^2\,d\mu = \int_{E_1} a|f|^2\,d\mu + \int_{E_2} a|f|^2\,d\mu \\ &= \int_X a|\chi_{E_1}f|^2\,d\mu + \int_X a|\chi_{E_2}f|^2\,d\mu \\ &= \int_X a|g_1|^2\,d\mu + \int_X a|g_2|^2\,d\mu \\ &= (Ag_1, g_1) + (Ag_2, g_2) \qquad \text{where } g_i(c) = \chi_{E_i}(x)f(x).\end{aligned}$$

But $\operatorname{Re}(Ag_i, g_i) = 0$ since Re $a(x) \geqq 0$ and $(Ag_1, g_1) + (Ag_2, g_2) = 0$. Also $\operatorname{Im}(Ag_1, g_1) \geqq 0$ and $\operatorname{Im}(Ag_2, g_2) \leqq 0$. Note that $(Ag_1, g_1) \neq 0$. For if $(Ag_1, g_1) = 0$ then $(Ag_2, g_2) = 0$ which implies that f and a have complementary support which is impossible since $\|f\| = 1$ and $|a(x)| > 0$ a.e. Thus $(Ag_2, g_2) \neq 0$. Set $h_i(x) = g_i(x)/\|g_i\|$. Let $\lambda_i = (Ah_i, h_i)$. Thus we have the two points λ_1, λ_2 in the numerical range, both on the imaginary axis, with 0 as an interior point of the line segment which joins them. This contradicts the assumption that 0 is an extreme point of the numerical range. Therefore 0 is an eigenvalue and the theorem is proved.

As an example consider the Hilbert space $L_2(X, \mu)$ where X is the closed unit disc and μ is the two-dimensional Lebesgue measure. Let A be the normal eigenvalue free operator

$$(Af)(z) = zf(z).$$

Since the spectrum of A is the closed unit disc and hence is the closure of the numerical range, without computation one may conclude that the numerical range is exactly the open unit disc.

UNIVERSITY OF MICHIGAN