# HOMEOMORPHIC CONJUGACY OF AUTOMORPHISMS ON THE TORUS[1]

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**Introduction.** Let $\gamma$ be a continuous map of the $n$-dimensional torus $T^n = R^n/Z^n$ into itself where $R^n$ is an $n$-dimensional real Euclidean space and $Z^n$ is the subgroup of $R^n$ with integral coordinates. Let $\pi: R^n \to T^n$ denote the universal covering map. There is a unique $c = (c_1, \cdots, c_n) \in R^n$ with $0 \leqq c_i < 1$, $i = 1, \cdots, n$, such that $\pi(c) = \gamma(0)$ and a unique continuous map $F: R^n \to R^n$ with $F(0) = c$ which is a "lifting" of $\gamma$, i.e., which satisfies $\pi F = \gamma\pi$. If we put $G(x) = F(x) - c$, $x \in R^n$, then $G|Z^n$ is a homomorphism of $Z^n$ into itself and therefore extends uniquely to a linear map $L: R^n \to R^n$. In fact making the canonical identification of $Z^n$ with the fundamental group $\pi_1(T^n)$ of $T^n$, $G|Z^n$ is just the homomorphism of $\pi_1(T^n)$ induced by $\gamma$. It follows that if $\gamma$ is a homeomorphism then $G|Z^n$ is an automorphism of $Z^n$ hence $L \in \mathrm{SL}(n, Z)$, the group of linear automorphisms of $R^n$ whose matrices are unimodular, i.e., have determinant $\pm 1$ and integer entries.

We next note that

$$F(x) = L(x) + P(x) + c \tag{1}$$

where $P: R^n \to R^n$ is a continuous periodic map (i.e., $P(x+\nu) = P(x)$, $x \in R^n$, $\nu \in Z^n$) satisfying $P(0) = 0$. This fact is established by considering $P(x) = F(x) - L(x) - c$. Clearly $P(0) = 0$. In view of the fact that $L\nu \in Z^n$ whenever $\nu \in Z^n$ we have $\pi(P(x+\nu) - P(x)) = \pi(F(x+\nu) - F(x) - L\nu) = \gamma(\pi x + \pi\nu) - \gamma(\pi x) - \pi L\nu = \gamma(\pi x) - \gamma(\pi x) = 0$; consequently $P(x+\nu) - P(x)$ is always in $Z^n$. Since $R^n$ is connected and $Z^n$ is discrete $P(x+\nu) - P(x)$ is (for $\nu$ fixed) independent of $x$. Thus $P(x+\nu) - P(x) = P(0+\nu) - P(0) = P(\nu) = F(\nu) - L\nu - c = G(\nu) - L\nu$ which is zero by definition $L|Z^n = G|Z^n$. We shall call $L$ the linear part, $P$ the periodic part, and $c$ the constant part of the lifting $F$ and when necessary place subscripts on these symbols to indicate the mapping on $T^n$ from which they came.

The linear, periodic, and constant part of a lifting are unique; for let $L'$, $P'$, $c'$, be other ones. $F(x) = L(x) + P(x) + c = L'(x) + P'(x) + c'$, yields $L(x) - L'(x) = P(x) - P'(x) + c - c'$. The right-hand side of the

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last relation is linear while the left is periodic. This can only occur if $L-L'=0$. Then since $P(0)=P'(0)=0$, it follows that $c=c'$ and $P=P'$.

If $\gamma$ is a continuous automorphism of $T^n$, then $F_\gamma$ is linear; thus its periodic and constant parts vanish so that $F_\gamma=L_\gamma$. Also every $L\in\mathrm{SL}(n, Z)$ is the lifting of a continuous automorphism on $T^n$. The correspondence $\gamma\rightarrow L_\gamma$ is an isomorphism of the group $\mathrm{Aut}(T^n)$ with $\mathrm{SL}(n, Z)$. More generally the mapping $\gamma\rightarrow L_\gamma$ is a homomorphism of the group $\mathrm{Homeo}(T^n)$ onto $\mathrm{SL}(n, Z)$. We shall prove this by showing that $L_{\alpha\beta}=L_\alpha L_\beta$ for any two homeomorphisms $\alpha$ and $\beta$ of $T^n$ onto itself. By uniqueness of lifting

$$F_{\alpha\beta} = F_\alpha F_\beta. \tag{2}$$

On one hand,

$$F_{\alpha\beta}(x) = L_{\alpha\beta}(x) + P_{\alpha\beta}(x) + c_{\alpha\beta}; \tag{3}$$

on the other hand, using (1) and adding and subtracting $P_\alpha(c_\beta)$,

$$\begin{aligned} F_\alpha F_\beta(x) = L_\alpha L_\beta(x) + [L_\alpha P_\beta(x) + P_\alpha(L_\beta(x) + P_\beta(x) + c_\beta) - P_\alpha(c_\beta)] \\ + L_\alpha(c_\beta) + P_\alpha(c_\beta) + c_\alpha. \end{aligned} \tag{4}$$

Because $L_\beta Z^n\subseteq Z^n$ the term in the brackets is periodic. This term vanishes when $x=0$ so that it is a periodic part of the lifting $F_{\alpha\beta}$. From the uniqueness of the various parts of a lifting

$$L_{\alpha\beta} = L_\alpha L_\beta, \tag{5}$$

$$P_{\alpha\beta}(x) = L_\alpha P_\beta(x) + P_\alpha(L_\beta(x) + P_\beta(x) + c_\beta) - P_\alpha(c_\beta), \tag{6}$$

$$c_{\alpha\beta} = L_\alpha(c_\beta) + P_\alpha(c_\beta) + c_\alpha. \tag{7}$$

Finally if $\gamma$ is a continuous automorphism of $T^n$, it preserves Haar measure on $T^n$ and to such transformations we can apply the notions of ergodic theory [1].

THEOREM. *If $\alpha$ and $\beta$ are continuous automorphisms of $T^n$ such that*

$$\gamma\alpha\gamma^{-1} = \beta \tag{8}$$

*where $\gamma$ is a homeomorphism of $T^n$ onto itself then*

(i) $L_\gamma L_\alpha L_\gamma^{-1}=L_\beta$ *($\alpha$ and $\beta$ are conjugate elements in the group of measure preserving transformations on $T^n$).*

(ii) *$c_\gamma$ is a fixed point of $L_\beta$ ($\gamma(0)$ is a fixed point of $\beta$).*

(iii) *If $\alpha$ is ergodic then $P_\gamma=0$ ($\gamma$ is a continuous automorphism of $T^n$ composed with a rotation. The rotation is by a fixed point of $\beta$ and the continuous automorphism satisfies the conjugacy relation between $\alpha$ and $\beta$).*

Proof. Relation (i) follows immediately from (5).

From (7)

$$c_{\gamma\alpha} = L_\gamma(c_\alpha) + P_\gamma(c_\alpha) + c_\gamma$$

and

$$c_{\beta\gamma} = L_\beta(c_\gamma) + P_\beta(c_\gamma) + c_\beta.$$

From (8) $\gamma\alpha=\beta\gamma$. Since the constant part of the lifting for this mapping is unique, $c_{\gamma\alpha}=c_{\beta\gamma}$.

Therefore because $P_\beta=0$ and $c_\alpha=c_\beta=0$, we have statement (ii) that $L_\beta(c_\gamma)=c_\gamma$.

It is convenient to prove (iii) in the following steps.

*Step* I. Define the function $Q$ by $Q(x)=L_\gamma^{-1}P_\gamma(x)$. Then $QL_\alpha^m=L_\alpha^mQ$ for all $m\in Z$. To prove this we first derive from (6) that $P_{\gamma\alpha}(x)=P_\gamma L_\alpha(x)$ and $P_{\beta\gamma}(x)=L_\beta P_\gamma(x)$. By hypothesis (8) and uniqueness of periodic part $P_\gamma L_\alpha=L_\beta P_\gamma$. Substituting (i) in this expression $P_\gamma L_\alpha=L_\gamma L_\alpha L_\gamma^{-1}P_\gamma$ so that $QL_\alpha=L_\alpha Q$. Thereupon Step I is obtained by induction on $m$.

*Step* II. We next recall that if $L: R^n\to R^n$ is an invertible linear operator and if $\{L^m x: m\in Z\}$ is bounded then either $x=0$ or $\{L^m x: m\in Z\}$ is bounded away from zero. One way to see this is to express $x$ in a basis that puts $L$ in Jordan form. It can be verified that the hypothesis $\{L^m x: m\in Z\}$ is bounded implies that $x$ is a linear combination of characteristic vectors of $L$ belonging to characteristic values of absolute value one.

*Step* III. If $\{L_\alpha^m Q(x): m\in Z\}$ is not bounded away from zero then $Q(x)=0$. This follows from Step II. The set $\{L_\alpha^m Q(x)=QL_\alpha^m(x): m\in Z\}$ is bounded because $Q$ being continuous and periodic is bounded.

*Step* IV. If $\{L_\alpha^m x: m\in Z\}$ is not bounded away from $Z^n$ then $Q(x)=0$. Since $Q$ is periodic, $L_\alpha^m Q(x)=L_\alpha^m(x)=Q(L_\alpha^m(x)-\nu)$ for $\nu\in Z^n$. By hypothesis there exists a subsequence $\{m_i: i=1, 2, \cdots\}$ of $Z$ and a subset $\{\nu_i: i=1, 2, \cdots\}$ of $Z^n$ such that $L_\alpha^{m_i}x-\nu_i\to 0$, $i\to\infty$. Since $Q$ is continuous and $Q(0)=0$, $L_\alpha^{m_i}Q(x)=Q(L_\alpha^{m_i}(x)-\nu_i)\to 0$. By Step III $Q(x)=0$.

Now $\alpha$ is ergodic and so almost all orbits of $\alpha$ are dense in $T^n$. In particular the zero element in $T^n$ is a limit point for almost all orbits or in other words $\{L_\alpha^m x: m\in Z\}$ is not bounded away from $Z^n$ for almost all $x\in R^n$. From Step IV $Q(x)=0$ almost everywhere. By continuity $Q=0$ and since $L_\gamma^{-1}$ is nonsingular $P_\gamma=0$.

**Remarks.** Questions arise whether the theorem holds under weaker hypotheses. In (iii) one cannot merely drop the assumption of ergodicity, for choosing $\alpha$ to be the identity transformation removes all restrictions on the homeomorphism $\gamma$. Another question is whether the theorem holds if $\gamma$ is a measure preserving transformation instead of a homeomorphism. A positive answer would be significant in ergodic theory, for then an example could be constructed of two Kolmogoroff transformations [2] with the same entropy but which are not conjugate.

## References

1. P. R. Halmos, *Lectures in ergodic theory*, Publ. Math. Soc. Japan No. 3, The Mathematical Society of Japan, Tokyo, 1956.

2. Ja. G. Sinaĭ, *Probabilistic ideas in ergodic theory*, Amer. Math. Soc. Transl. (2) **31** (1963), 62–84.

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