References

1. L. Auslander, *Bierberbach's theorem on space groups and discrete uniform subgroups of Lie groups*. II, Amer. J. Math. **83** (1961), 276–280.

2. L. Bieberbach, *Über die Bewegungsgruppen der Euklidischen Raüme*. I, Math. Ann. **70** (1911), 297–336.

3. ———, *Über die Bewegungsgruppen der Euklidischen Raüme*. II, Math. Ann. **72** (1912), 400–412.

4. S. Helgason, *Differential geometry and symmetric spaces*, Academic Press, New York, 1962.

Purdue University and
University of California, Berkeley

EXTENSION OF NORMAL FAMILIES OF HOLOMORPHIC FUNCTIONS

THEODORE J. BARTH[1]

Let X be a Stein manifold, and let A be an analytic subset of X A well-known application of Cartan's Theorem B [2, Théorème 3 p. 52] states that each holomorphic function on A is the restriction of a holomorphic function on X. This paper presents a generalization of this application, namely that each normal family of holomorphic functions on A is the restriction of a normal family of holomorphic functions on X.

1. Let X be a topological space which is σ-compact, i.e., the union of a countable family of compact sets. Let $K(X)$ denote the set of all compact subsets of X. For $K \in K(X)$ and $f: X \to \mathbf{C}$ define $\|f\|_K = \sup\{|f(x)| \mid x \in K\}$. Define

$$B(X) = \{f \mid f: X \to \mathbf{C}, \|f\|_K < \infty \text{ for all } K \in K(X)\}.$$

Clearly $B(X)$ is a complex vector space, and $\{\| \; \|_K \mid K \in K(X)\}$ is a family of pseudonorms on $B(X)$ which then becomes a locally convex vector space. Since X is σ-compact, $B(X)$ is metrizable, and it is readily checked to be a Fréchet space.

Definition. Let V be a vector subspace of $B(X)$. We say that a set $F \subset V$ is normal with respect to V iff every sequence in F has a subsequence which converges in V.

Received by the editors October 29, 1964.

[1] This research supported in part by the National Science Foundation while the author held a Graduate Fellowship at the University of Notre Dame.

LEMMA 1. *Let V be a vector subspace of $B(X)$. Then $F \subset V$ is normal with respect to V if and only if $\overline{F}$ is a compact subset of V.*

We say that $F \subset B(X)$ is uniformly bounded on compact sets iff $\sup\{\|f\|_K \mid f \in F\} < \infty$ for all $K \in K(X)$. If V is a vector subspace of $B(X)$ and $F \subset V$ is normal with respect to V, then F is uniformly bounded on compact sets.

DEFINITION. Let V be a vector subspace of $B(X)$. We say that the Vitali theorem holds for V iff every set $F \subset V$ which is uniformly bounded on compact sets is normal with respect to V.

LEMMA 2. *Let V be a vector subspace of $B(X)$. If the Vitali theorem holds for V, then V is a closed subset of $B(X)$, i.e., V is a Fréchet space with the induced psudonorms.*

PROOF. Let $f \in \overline{V}$. Then there exists a sequence $\{f_n\}$ in V such that $f_n \to f$ for $n \to \infty$. Take $K \in K(X)$. There exists an integer n_0 such that $\|f_n - f\|_K < 1$ for $n \geqq n_0$. Let $M = \max\{\|f_n\|_K + 1 \mid n = 1, \cdots, n_0\}$. Then $\|f_n\|_K \leqq M$ for $n = 1, 2, \cdots$. Hence $\sup\{\|f_n\|_K \mid n = 1, 2, \cdots\} \leqq M < \infty$, that is $\{f_n \mid n = 1, 2, \cdots\}$ is uniformly bounded on compact sets. Since the Vitali theorem holds for V, $\{f_n\}$ has a subsequence which converges in V. The limit of this subsequence must be f. Hence $f \in V$.

LEMMA 3. *Let A be a closed subset of the σ-compact space X. Define $R: B(X) \to B(A)$ by $R(f) = f \mid A$ for $f \in B(X)$. Then R is continuous and linear.*

PROOF. Clearly R is linear. Each compact subset of A is also compact in X. Hence each pseudonorm in $B(A)$ is the restriction of a pseudonorm in $B(X)$. Therefore R is continuous.

LEMMA 4. *Let E and F be Fréchet spaces, and let $u: E \to F$ be continuous, linear, and surjective. Let K be a compact subset of F. Then there exists a compact subset K' of E such that $u(K') = K$.*

PROOF. Let $G = u^{-1}(0)$. Then G is a closed linear subspace of E, hence E/G is a Fréchet space. Let $r: E \to E/G$ be the residual map, and define $v: E/G \to F$ by $v \circ r = u$. Then v is continuous, linear, and bijective. According to [**4**, Satz (1), p. 170], v is a topological isomorphism. The lemma then follows from [**4**, Satz (7), p. 281].

2. If X is a complex space, we define $H(X)$ to be the set of holomorphic functions on X. As has been proved by Gunning [**3**] and Andreotti and Stoll [**1**, pp. 326–327], the Vitali theorem holds for

$H(X)$. We say that $F \subset H(X)$ is a normal family of holomorphic functions iff F is normal with respect to $H(X)$.

THEOREM. *Let X be a Stein manifold, and let A be an analytic subset of X. Let $F = \{f_\lambda \mid \lambda \in \Lambda\}$ be a normal family of holomorphic functions on A. Then there exists a normal family $G = \{g_\lambda \mid \lambda \in \Lambda\}$ of holomorphic functions on X such that $g_\lambda \mid A = f_\lambda$ for $\lambda \in \Lambda$.*

PROOF. According to the well-known application of Theorem B of Cartan referred to at the beginning of this paper, the restriction map $R: H(X) \rightarrow H(A)$, defined by $R(f) = f \mid A$, is onto. By Lemma 3, R is continuous and linear. Since the Vitali theorem holds for $H(X)$ and $H(A)$, they are Fréchet spaces (Lemma 2). Application of Lemma 1 and Lemma 4 completes the proof.

REFERENCES

1. A. Andreotti and W. Stoll, *Extension of holomorphic maps*, Ann. of Math. **72** (1960), 312–349.

2. H. Cartan, *Variétés analytiques complexes et cohomologie*, Colloque sur les fonctions de plusieurs variables, held at Brussels, 1953, pp. 41–55, Thone, Liége, 1953.

3. R. C. Gunning, *On Vitali's Theorem for complex spaces with singularities*, J. Math. Mech. **8** (1959), 133–141.

4. G. Köthe, *Topologische lineare Räume*. I, Springer-Verlag, Berlin, 1960.

UNIVERSITY OF NOTRE DAME