THE RAPIDITY OF CONVERGENCE OF THE HERMITE-
FEJÉR APPROXIMATION TO FUNCTIONS
OF ONE OR SEVERAL VARIABLES

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1. The Hermite-Fejér polynomials $H_n(f, x)$ are important approximating polynomials to a given real function $f(x)$ defined on $[-1, 1]$. For every positive integer $n$,

$$(1) \quad H_n(f, x) = \sum_{k=1}^{n} f(x_k^{(n)}) A_k^{(n)}(x)$$

where

$$A_k^{(n)}(x) = (1 - xx_k^{(n)}) \left[ T_n(x) / \left( n(x - x_k^{(n)}) \right) \right]^2 \quad (k = 1, 2, \ldots, n),$$

$$T_n(x) = 2^{n-1} \prod_{k=1}^{n} (x - x_k^{(n)}), \quad x_k^{(n)} = \cos \left( \frac{2k - 1}{2n} \pi \right) \quad (k = 1, 2, \ldots, n).$$

For $n = 1, 2, \ldots$, $T_n(x)$ (the $n$th degree Tchebycheff polynomial of the first kind) satisfies $T_n(\cos \theta) = \cos (n\theta)$, and we also have

$$(2) \quad A_k^{(n)}(x) \geq 0 \quad \text{for } k = 1, 2, \ldots, n \text{ and every } x \in [-1, 1],$$

$$(3) \quad \sum_{k=1}^{n} A_k^{(n)}(x) = 1,$$

$$(4) \quad H_n(f, x_k^{(n)}) = f(x_k^{(n)}) \quad (k = 1, 2, \ldots, n),$$

$$(5) \quad H_n(f, x_k^{(n)}) = 0 \quad (k = 1, 2, \ldots, n).$$

2. Suppose $f$ is a real function, continuous in $[-1, 1]$. Then a classical result of Fejér [1] states that $H_n(f, x)$ converges uniformly to $f(x)$ on $[-1, 1]$. As to the rapidity of convergence, E. Moldovan published in [2] the estimate $\max_{-1 \leq x \leq 1} |f(x) - H_n(f, x)| \leq 2\pi \omega(n^{-1} \log n)$, where $\omega$ is the modulus of continuity of $f$ in $[-1, 1]$.

3. One of our purposes in this paper is to construct an analog of the polynomial (1) for functions $f$ of several variables and to study the corresponding rapidity of convergence. Since the Romanian paper

Presented to the Society, April 13, 1965 under the title Hermite-Fejér polynomials for functions of several variables; received by the editors November 18, 1964.
is inaccessible to many readers and is also wanting, we shall prove in Theorem 2 a result which is essentially the same as that published by Moldovan. Theorems 1 and 2 will be used to obtain Theorems 3 and 4 concerning functions of several variables.

4. Theorem 1. Let $f$ be a real function satisfying throughout $[-1, 1]$

$$|f(v) - f(u)| \leq \lambda |v - u|$$

where $\lambda$ is a positive constant. Then for $n = 1, 2, \cdots$ and every $x \in [-1, 1]$,

$$|f(x) - H_n(f, x)| < 4\pi n^{-1}(\alpha + \log n),$$

where $\alpha = \frac{1}{e} + C - \log 2 = 0.384 \cdots$, $C$ being Euler's constant.

Proof. Let $n$ be a positive integer and let $-1 \leq x \leq 1$. We shall prove (6). Let $x = \cos \theta$ ($0 \leq \theta \leq \pi$), and let $\theta_k = ((2k - 1)/2n)\pi$ ($k = 1, 2, \cdots, n$). Since by (4), $H_n(f, \cos \theta_k) = f(\cos \theta_k)$ for $k = 1, 2, \cdots, n$, we may assume $\theta \neq \theta_k$, $k = 1, 2, \cdots, n$. By (3), (1), and (2) $|f(x) - H_n(f, x)|$

$$= \left| \sum_{k=1}^{n} [f(x) - f(x_k^n)]A_k^{(n)}(x) \right| \leq \sum_{k=1}^{n} |f(x) - f(x_k^n)| A_k^{(n)}(x)$$

$$= \sum_{k=1}^{n} |f(x) - f(x_k^n)| (1 - xx_k^n) \left[ T_n(x)/\{n(x - x_k^n)\} \right]^2$$

$$\leq \lambda \sum_{k=1}^{n} |x - x_k^n| (1 - xx_k^n) \left[ T_n(x)/\{n(x - x_k^n)\} \right]^2$$

$$= \lambda n^{-2} \sum_{k=1}^{n} |\cos \theta - \cos \theta_k| (1 - \cos \theta \cos \theta_k) [\cos (n\theta)/(\cos \theta - \cos \theta_k)]^2$$

$$< \lambda n^{-2} \sum_{k=1}^{n} |\theta - \theta_k| (1 - \cos \theta \cos \theta_k) [\cos (n\theta)/(\cos \theta - \cos \theta_k)]^2$$

$$\leq \lambda n^{-2} \sum_{k=1}^{n} |\theta - \theta_k| [1 - \cos \theta \cos \theta_k + \sin \theta \sin \theta_k]$$

$$\cdot [(\cos (n\theta) - \cos (n\theta_k))/(\cos \theta - \cos \theta_k)]^2$$

$$= \lambda n^{-2} \sum_{k=1}^{n} |\theta - \theta_k| 2 \sin^2 \{\theta + \theta_k)/2\} \sin^2 \{n(\theta + \theta_k)/2\}$$

$$\cdot \sin^2 \{n(\theta - \theta_k)/2\} \sin^2 \{\theta + \theta_k)/2\} \sin^2 \{n(\theta - \theta_k)/2\}$$

$$< 2\lambda n^{-2} \sum_{k=1}^{n} |\theta - \theta_k| [\sin^2 \{n(\theta - \theta_k)/2\}/\sin^2 \{\theta - \theta_k)/2\}]^2.$$
Suppose $n \geq 4$ and $\theta_j < \theta < \theta_{j+1}$, $2 \leq j \leq n - 2$. Since $|\sin(ny)/\sin y| < n$ for every real $y \neq 0, \pm \pi, \pm 2\pi, \cdots$, therefore

$$
|\theta - \theta_j| \left[ \frac{\sin\left\{ n(\theta - \theta_j)/2 \right\}}{\sin\left\{ (\theta - \theta_j)/2 \right\}} \right]^2
\quad + \quad |\theta - \theta_{j+1}| \left[ \frac{\sin\left\{ n(\theta - \theta_{j+1})/2 \right\}}{\sin\left\{ (\theta - \theta_{j+1})/2 \right\}} \right]^2
\quad < \quad |\theta - \theta_j| n^2 + |\theta - \theta_{j+1}| n^2 = n\pi.
$$

Since $y/\sin(y/2)$ is strictly increasing in $(0, \pi)$, we have $y/\sin(y/2) < \pi/\sin(\pi/2) = \pi$ whenever $0 < y < \pi$. For such $y$ one has

$$
y\left[ \frac{\sin(ny/2)}{\sin(y/2)} \right]^2 = y^{-1}[y/\sin(y/2)]^2 \sin^2(ny/2) < \pi^2/y.
$$

For $k = 1, 2, \cdots, j - 1, \theta - \theta_k > (j-k)\pi/n$, and so

$$(\theta - \theta_k) \left[ \frac{\sin\left\{ n(\theta - \theta_k)/2 \right\}}{\sin\left\{ (\theta - \theta_k)/2 \right\}} \right]^2 < \pi^2/\theta - \theta_k < n\pi/(j-k).$$

Consequently,

$$
\sum_{k=1}^{j-1} |\theta - \theta_k| \left[ \frac{\sin\left\{ n(\theta - \theta_k)/2 \right\}}{\sin\left\{ (\theta - \theta_k)/2 \right\}} \right]^2
\quad < \quad n\pi \sum_{k=1}^{j-1} (j-k)^{-1} = n\pi \sum_{k=1}^{j-1} k^{-1}.
$$

Similarly,

$$
\sum_{k=j+2}^{n} |\theta - \theta_k| \left[ \frac{\sin\left\{ n(\theta - \theta_k)/2 \right\}}{\sin\left\{ (\theta - \theta_k)/2 \right\}} \right]^2
\quad < \quad n\pi \sum_{k=j+2}^{n} [k - (j + 1)]^{-1} = n\pi \sum_{k=1}^{n-1-j} k^{-1}.
$$

From (8), (7) and (9) we obtain

$$
\sum_{k=1}^{n} |\theta - \theta_k| \left[ \frac{\sin\left\{ n(\theta - \theta_k)/2 \right\}}{\sin\left\{ (\theta - \theta_k)/2 \right\}} \right]^2
\quad < \quad n\pi \left( 1 + \sum_{k=1}^{j-1} k^{-1} + \sum_{k=1}^{n-1-j} k^{-1} \right),
$$

and therefore

$$
|f(x) - H_n(f, x)| < 2\lambda\pi n^{-1} \left( 1 + \sum_{k=1}^{j-1} k^{-1} + \sum_{k=1}^{n-1-j} k^{-1} \right).
$$

If $n$ is even, then
\[
\sum_{k=1}^{n-1} k^{-1} + \sum_{k=1}^{n-2} k^{-1} \leq \max_{1 \leq s \leq n-3} \left( \sum_{k=1}^{s} k^{-1} + \sum_{k=1}^{n-2-s} k^{-1} \right) = 2[1+2^{-1} + \cdots + \{(n-2)/2\}^{-1}] = 2[1+2^{-1} + \cdots + \{(n-3)/2\}^{-1} + \{(n-1)/2\}^{-1}].
\]

If \( n \) is odd, then
\[
\sum_{k=1}^{n-1} k^{-1} + \sum_{k=1}^{n-2} k^{-1} \leq \max_{1 \leq s \leq n-3} \left( \sum_{k=1}^{s} k^{-1} + \sum_{k=1}^{n-2-s} k^{-1} \right) = 2[1+2^{-1} + \cdots + \{(n-3)/2\}^{-1} + \{(n-1)/2\}^{-1}].
\]

Thus
\[
|f(x) - H_n(f, x)| < 2\lambda n^{-1}(1+2[1+2^{-1} + \cdots + \{(n-2)/2\}^{-1}])
\]

if \( n \) is even,
\[
|f(x) - H_n(f, x)| < 2\lambda n^{-1}(1+2[1+2^{-1} + \cdots + \{(n-3)/2\}^{-1} + \{(n-1)/2\}^{-1})
\]

if \( n \) is odd.

One can show similarly, that (11) (with \( n \geq 4 \)) is true for every other position of \( \theta \) in \([0, \pi]\).

Since \( \sum_{k=1}^{s} k^{-1} - \log q < C \) for \( q = 2, 3, 4, \ldots \), the desired inequality (6) follows from (11) for \( n \) even and \( \geq 4 \).

Consider the sequence
\[
a_q = \left( \sum_{k=1}^{q-1} k^{-1} \right) + (2q)^{-1} - \log[(2q+1)/2] \quad (q = 2, 3, 4, \cdots).
\]

For every \( q \geq 2 \) we have
\[
a_{q+1} - a_q = \frac{1}{2}[q^{-1} + (q+1)^{-1}] - \log[1+2/(2q+1)] > \frac{1}{2}[q^{-1} + (q+1)^{-1}] - 2/(2q+1) > 0.
\]

Since \( \lim_{q \to \infty} a_q = C \), we have \( a_q < C \) (\( q = 2, 3, 4, \cdots \)).

Suppose \( n \) is odd and \( > 4 \). Setting \( q = (n-1)/2 \) we have \( C > a_q = 1+2^{-1} + \cdots + \{(n-3)/2\}^{-1} + (n-1)^{-1} - \log(n/2), \) and (6) follows from (11).

Finally one can verify (6) for \( n = 1, 2, 3 \) by the same sort of calculations which led to (10).

5. Theorem 2. Let \( f \) be a real function, defined and bounded on \([-1, 1]\). For every \( \delta \in [0, 2] \) let
\[
\omega(\delta) = \sup_{-1 \leq v, w \leq 1, |v-w| \leq \delta} |f(v) - f(w)|.
\]
Then for \( n = 2, 3, 4, \cdots \) we have
\[
\sup_{-1 \leq x \leq 1} |f(x) - H_n(f, x)| \leq [2 + 4\pi + \epsilon_n]\omega(\pi^{-1}\log n)
\]
where \( \epsilon_n \) depends on \( n \) only and \( \epsilon_n \to 0 \) as \( n \to \infty \).

**Proof.** Let \( n \) be a positive integer. Let \( x_0, x_1, \cdots, x_N \ (N \geq 1) \) be reals with \(-1 = x_0 < x_1 < \cdots < x_N = 1\). Let \( f_n \) be the function with domain \([-1, 1]\) such that \( f_n(x_k) = f(x_k) \), \( k = 0, 1, \cdots, N \) and such that \( f_n \) is linear in each \([x_{k-1}, x_k]\) \((k = 1, 2, \cdots, N)\). If \( x_{k-1} \leq u < v \leq x_k \) for some \( k \), then
\[
|f_n(v) - f_n(u)| = |f(x_k) - f(x_{k-1})| / (x_k - x_{k-1}) \leq \omega(x_k - x_{k-1}) / (x_k - x_{k-1}) \leq \lambda
\]
where
\[
\lambda = \max_{1 \leq k \leq N} \left[ \omega(x_k - x_{k-1}) / (x_k - x_{k-1}) \right].
\]
Therefore \( |f_n(v) - f_n(u)| \leq \lambda (v - u) \) whenever \(-1 \leq u < v \leq 1\). By Theorem 1 we have throughout \([-1, 1]\),
\[
|f_n(x) - H_n(f_n, x)| \leq 4\lambda \pi n^{-1}(\alpha + \log n).
\]
One easily verifies that throughout \([-1, 1]\),
\[
|f(x) - f_n(x)| \leq \mu
\]
where \( \mu = \max_{1 \leq k \leq N} \omega(x_k - x_{k-1}) \). Therefore, by (1), (2) and (3) we have throughout \([-1, 1]\),
\[
|H_n(f, x) - H_n(f_n, x)| \leq \sum_{k=1}^{n} |f(x_k^{(n)}) - f_n(x_k^{(n)})| A_k^{(n)}(x) \leq \mu.
\]
From (12), (13) and (14) we get
\[
\sup_{-1 \leq x \leq 1} |f(x) - H_n(f, x)| \leq 2\mu + 4\lambda \pi n^{-1}(\alpha + \log n)
\]
and so
\[
\sup_{-1 \leq x \leq 1} |f(x) - H_n(f, x)|
\]
\[
\leq \inf_{-1 = x_0 < x_1 < \cdots < x_N = 1; N=1,2,3,\cdots} \left\{ 2 \max_{1 \leq k \leq N} \omega(x_k - x_{k-1}) + 4 \max_{1 \leq k \leq N} \left[ \omega(x_k - x_{k-1}) / (x_k - x_{k-1}) \right] \pi n^{-1}(\alpha + \log n) \right\}.
\]
If we take in particular for some positive integer \( N \), \( x_k = -1 + (2k/N) \) \((k = 0, 1, \ldots, N)\), then from (15) we obtain

\[
(16) \quad \sup_{-1 \leq x \leq 1} |f(x) - H_n(f, x)| \leq 2\omega(2/N)[1 + \pi N n^{-1}(\alpha + \log n)].
\]

Thus

\[
\sup_{-1 \leq x \leq 1} |f(x) - H_n(f, x)| \leq \inf_{N=1,2,\ldots} \{2\omega(2/N)[1 + \pi N n^{-1}(\alpha + \log n)]\}. 
\]

Suppose \( n \geq 2 \), and let \( N \) be the integral part of \( 1 + (2k/\log n) \). Then (16) yields

\[
(17) \quad \sup_{-1 \leq x \leq 1} |f(x) - H_n(f, x)| \leq [2 + 4\pi + \epsilon_n] \omega(n^{-1} \log n),
\]

where \( \epsilon_n = 2\pi n^{-1}[\alpha + \log n + (2\pi n/\log n)] \).

6. Consider now a real function \( f(x_1, x_2, \ldots, x_p) \) defined for \(-1 \leq x_k \leq 1, k = 1, 2, \ldots, p\). Let \( n_1, n_2, \ldots, n_p \) be positive integers and set

\[
(18) \quad H_{n_1, \ldots, n_p}(f, x_1, x_2, \ldots, x_p) = \sum_{k=1}^{n_1} \cdots \sum_{h_p=1}^{n_p} f(x_h^{(n_1)}(x_1), \ldots, x_h^{(n_p)}(x_p)) A^{(n_1)}_h(x_1) \cdots A^{(n_p)}_h(x_p).
\]

The polynomial (18), which reduces to an Hermite-Fejér polynomial when \( p = 1 \), has properties analogous to (4) and (5). Thus, for \( k = 1, 2, \ldots, p \), let \( j_k \) be a positive integer \( \leq n_k \). Then a repeated application of (4) yields easily

\[
H_{n_1, \ldots, n_p}(f, x_1, x_2, \ldots, x_p) = f(x_{j_1}^{(n_1)}(x_1), \ldots, x_{j_p}^{(n_p)}(x_p)).
\]

Also, if \( p \geq 2 \), and if \( 1 \leq k \leq p, 1 \leq j \leq n_k \), then

\[
(19) \quad \left( \frac{\partial}{\partial x_k} H_{n_1, \ldots, n_p}(f, x_1, \ldots, x_p) \right)_{x_k=\gamma_i^{(n_k)}} = 0.
\]

Indeed, the left-hand side of (19) is identically equal to

\[
\sum_{h_p=1,2,\ldots,n_p} \prod_{s=1}^p A^{(n_s)}_{h_s}(x_s) \left( \frac{\partial}{\partial x_k} \sum_{h_1=1}^{n_1} \cdots \sum_{h_p=1}^{n_p} f(x_h^{(n_1)}(x_1), \ldots, x_h^{(n_p)}(x_p)) A^{(n_h)}_{h_1}(x_1) \cdots A^{(n_h)}_{h_p}(x_p) \right)_{x_k=\gamma_i^{(n_k)}}
\]

which is \( = 0 \) by (1) and (5).
From (3) one gets
\[ \sum_{k_1=1}^{n_1} \cdots \sum_{k_p=1}^{n_p} A_{h_1}^{(n_1)}(x_1) \cdots A_{h_p}^{(n_p)}(x_p) = 1. \]
Consequently, by virtue of (18), if \( p \geq 2 \),
\begin{align*}
&f(x_1, \cdots, x_p) - H_{n_1, \ldots, n_p}(f, x_1, \cdots, x_p) \\
= &\sum_{k_1=1}^{n_1} \cdots \sum_{k_p=1}^{n_p} \left[ f(x_1, \cdots, x_p) - f(x_1^{(n_1)}, \cdots, x_p^{(n_p)}) \right] \\
&\cdot A_{h_1}^{(n_1)}(x_1) \cdots A_{h_p}^{(n_p)}(x_p) \\
= &\sum_{k_1=1}^{n_1} \cdots \sum_{k_p=1}^{n_p} \sum_{r=1}^{p} \left[ f(x_1^{(n_1)}, \cdots, x_{r-1}^{(n_{r-1})}, x_r^{(n_r)}, \cdots, x_p^{(n_p)}) \right] \\
&- f(x_1^{(n_1)}, \cdots, x_{r-1}^{(n_{r-1})}, x_r^{(n_r)}, \cdots, x_p^{(n_p)}) \prod_{s=1}^{p} A_{h_s}^{(n_s)}(x_s) \\
= &\sum_{r=1}^{p} \sum_{k_{p+1-1, \ldots, n_q} = 1}^{n_r} \sum_{k_q=1}^{n_q} \left\{ \sum_{k_{r-1}^{(n_{r-1})}}^{n_{r-1}} \left[ f(x_1^{(n_1)}, \cdots, x_{r-1}^{(n_{r-1})}, x_r^{(n_r)}, \cdots, x_p^{(n_p)}) \right] \right\} \\
&- f(x_1^{(n_1)}, \cdots, x_{r-1}^{(n_{r-1})}, x_r^{(n_r)}, \cdots, x_p^{(n_p)}) \prod_{s=1}^{p} A_{h_s}^{(n_s)}(x_s).
\end{align*}

Here and below \( f(x_1^{(n_1)}, \cdots, x_{h_{r-1}}^{(n_{r-1})}, x_r^{(n_r)}, \cdots, x_p^{(n_p)}) \) means \( f(x_1, \cdots, x_p) \) if \( r = 1 \), and \( f(x_1^{(n_1)}, \cdots, x_{h_r}^{(n_r)}, x_{r+1}^{(n_{r+1})}, \cdots, x_p^{(n_p)}) \) means \( f(x_1^{(n_1)}, \cdots, x_{h_p}^{(n_p)}) \) if \( r = p \).

7. Theorem 3. Let \( f(x_1, x_2, \cdots, x_p) \) \((p \geq 2)\) be a real function defined for \(-1 \leq x_k \leq 1, k = 1, 2, \cdots, p\). For \( r = 1, 2, \cdots, p \), let \( \lambda_r \) be a positive constant such that
\[ |f(x_1, \cdots, x_{r-1}, v, x_{r+1}, \cdots, x_p) - f(x_1, \cdots, x_{r-1}, u, x_{r+1}, \cdots, x_p)| \leq \lambda_r |v - u| \]
whenever the \( x_q, u, \) and \( v \) are all in \([-1, 1]\). Let \( n_1, n_2, \cdots, n_p \) be positive integers. Then
\begin{align*}
|f(x_1, x_2, \cdots, x_p) - H_{n_1, n_2, \ldots, n_p}(f, x_1, x_2, \cdots, x_p)| \\
< \sum_{r=1}^{p} 4\lambda_r \pi n_r^{-1}(\alpha + \log n_r)
\end{align*}
throughout the cube \(-1 \leq x_k \leq 1, k = 1, 2, \cdots, p\).

Proof. Let \( x_1, x_2, \cdots, x_p \) be points of \([-1, 1]\). We shall prove (20). For \( r = 1, 2, \cdots, p \), we have by (3), (1), and by Theorem 1,
\[
\sum_{b=1}^{n_r} [f(x_{a1}^{(n_1)}, \ldots, x_{a_{r-1}}^{(n_{r-1})}, x_r, \ldots, x_p) \\
- f(x_{a1}^{(n_1)}, \ldots, x_{a_r}^{(n_r)}, x_{r+1}, \ldots, x_p)] A_{a_r}^{(n_r)}(x_r) \]
\leq 4\lambda_r n_r^{-1}(\alpha + \log n_r).
\]

Since for \(r = 1, 2, \ldots, p\),

\[
\sum_{b=1}^{n_r} \prod_{s=1; s \neq r}^{p} A_{a_s}^{(n_s)}(x_s) = 1
\]

and each summand is \(\geq 0\), therefore by the last paragraph of section 6, (20) holds.

Similarly, from Theorem 2 we obtain the following

**Theorem 4.** Let \(f(x_1, x_2, \ldots, x_p) \ (p \geq 2)\) be a real function defined and bounded for \(-1 \leq x_1, \ldots, x_k \leq 1, k = 1, 2, \ldots, p\). For every \(\delta \subseteq [0, 2]\) and every \(r \ (= 1, 2, \ldots, p)\) let

\[
\omega_r(\delta) = \sup |f(x_1, \ldots, x_{r-1}, v, x_{r+1}, \ldots, x_p) \\
- f(x_1, \ldots, x_{r-1}, u, x_{r+1}, \ldots, x_p)|,
\]

where \(x_1, \ldots, x_{r-1}, x_{r+1}, \ldots, x_p, u, v\) vary in \([-1, 1]\) with \(0 \leq v-u \leq \delta\). Let \(n_1, n_2, \ldots, n_p\) be positive integers \(\geq 2\). Then

\[
\sup_{-1 \leq x_k \leq 1} |f(x_1, x_2, \ldots, x_p) - H_{n_1, n_2, \ldots, n_p}(f, x_1, x_2, \ldots, x_p)|
\]

\leq \sum_{r=1}^{p} [2 + 4\pi + \epsilon_r(n_r^{-1} \log n_r)],

where each \(\epsilon_n \ (n = 2, 3, \ldots)\) depends on \(n\) only, and \(\epsilon_n \to 0\) as \(n \to \infty\).

**References**


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