The proof is completed by observing that

$$\int_0^1 \Omega(kf)\,d\mu \geqq \sum \Phi(2c_{m_n})\mu(E_{m_n}) \geqq \sum_{n=1}^{\infty} \alpha s_0.$$

Reference

1. Krasnosel'skii and Rutickii, *Convex functions and Orlicz spaces*, P. Noordhoff Ltd., Groningen, 1961.

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ON A COMBINATORIAL PROBLEM OF ERDÖS

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Let $C(n, m)$ denote the binomial coefficient $n!/(m!n-m!)$. Let S be a set containing N elements and let X be a collection of subsets of S with the property that if A, B and C are distinct elements of X, then $A\cup B\neq C$. Erdös [1], [2], has conjectured that X contains at most $KC(N, [N/2])$ elements where K is a constant independent of X and N. The problem is related to a result of Sperner [3] to the effect that if the collection X has the more restrictive property that no element of X contains any other, then X can have at most $C(N, [N/2])$ elements.

We show below that Erdös' conjecture for $K=2^{3/2}$ can be deduced directly from Sperner's result.

Let L_N be defined by

$$L_N \equiv 2^{[N/2]}C(N - [N/2], [\tfrac{1}{2}(N - [N/2])]) + 2^{N-[N/2]}C([N/2], [N/4]).$$

An easy calculation shows that L_N is always less than $2^{3/2}C(N, [N/2])$ to which it is asymptotic for large N. We prove:

Theorem. *If X is a family of subsets of an N element set S such that no three distinct A, B, C in X satisfy $A\cup B=C$, then X has less than L_N elements.*

Proof. For any finite set T and family X of subsets of T define

$$m_T(X) \equiv \{A \in X \mid B \in X \text{ and } B \subset A \text{ imply } B = A\}.$$

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Note that $m_T(X)$ satisfies the hypothesis of Sperner's theorem and hence $m_T(X)$ contains at most $C(M, [M/2])$ where M is the number of elements in T.

Let $S=T_1\cup T_2$ where $T_1\cap T_2=\emptyset$ and T_1 contains $[N/2]$ elements. For each subset $A\subset S$ let

$$D_j(A) = \{B \in X \mid B \cap T_j = A \cap T_j\}, \qquad j = 1, 2.$$

Note that $m_s(D_2(A))$ and $m_{T_1}(\{B\cap T_1 \mid B\in D_2(A)\})$ have the same number of elements. In consequence, since T_1 has $[N/2]$ elements, $m_s(D_2(A))$ can have at most $C([N/2], [N/4])$ elements. Similarly $m_s(D_1(A))$ can have at most $C(N-[N/2], [\frac{1}{2}(N-[N/2])])$ elements.

Next we show that if $A\in X$ then $A\in m_s(D_1(A))\cup m_s(D_2(A))$. Suppose $A\in X$ and $A\notin m_s(D_1(A))\cup m_s(D_2(A))$. Then there are subsets B_1 and B_2 such that $B_j\cap T_j=A\cap T_j$, $B_j\neq A$, $B_j\subset A$, $B_j\in X$, $j=1, 2$. But then, $B_1\cup B_2=A$ and B_1 and B_2 and A are distinct and hence $A\notin X$. Thus we have shown that $X\subset \bigcup_{A\in X} \{(m(D_1(A))\cup m(D_2(A))\}$.

Note that $m_s(D_1(A))=m_s(D_1(B))$ if $A\cap T_1=B\cap T_1$. Hence there are at most $2^{[N/2]}$ distinct families $m_s(D_1(A))$, one for each distinct $A\cap T_1$. Similarly, there are at most $2^{N-[N/2]}$ distinct families $m_s(D_2(A))$. Hence the number of elements in X is at most L_N. L_N can be reduced by $C([N/2], [N/4])\cdot C(N-[N/2], [\frac{1}{2}(N-[N/2])])$ by taking into account the overlap between the elements of the

$$m_s(D_1)\text{'s} \quad \text{and} \quad m_s(D_2)\text{'s}.$$

The proof above makes use of only part of the hypothesis; namely, that X contains no subset A which is a union of two others, B and C, with

$$B \cap T_1 = A \cap T_1,$$
$$C \cap (S - T_1) = A \cap (S - T_1),$$

for a given $[N/2]$ element subset T_1 of S. One can construct an X satisfying these conditions with only $2C([N/2], [N/4])C(N-[N/2], \frac{1}{2}[N-[N/2]])-1$ elements fewer than the maximum noted above, so that $2^{3/2}C(N, [N/2])(1+o(N))$ is a best bound, for families X subject to this weaker restriction.

The upper limit $2^{3/2}$ deduced for K above is not a best estimate under the more general limitation on X suggested by Erdös. If we use the fact that the intersections with T_1 of the elements of the D_2's must form a family satisfying our hypotheses for the $[N/2]$ element set T_1, the estimate for K given above can be reduced by approximately 5 percent for large N. The best value for K is probably 2 (realized for $N=1$) and, if the maximum number of elements of X is written as $K_N C(N, [N/2])$ it may be that K_N approaches as N increases.

The result may be straightforwardly extended to collections X restricted such that no element contains the union of j others. One can deduce $j^{3/2}C(N, [N/2])$ as upper limit on the number of elements in such an X.

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Bibliography

1. P. Erdös, *On a lemma of Littlewood and Offord*, Bull. Amer. Math. Soc. **51** (1945), 898–902.

2. S. Ulam, *A collection of mathematical problems*, Interscience, New York, 1960, p. 27.

3. E. Sperner, *Ein Satz über Untermengen einer endlichen Menge*, Math. Z. **27** (1928), 544–548.

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