EXTENSION OF FUNCTIONALS AND INEQUALITIES ON AN ABELIAN SEMI-GROUP[1]

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Let $(G, +)$ be an abelian semi-group and N a real function on G satisfying

(1) $N \geqq 0$,

(2) $N(x+y) \leqq N(x)N(y)$, for all $x, y \in G$.

The function $N^*(x) = \lim_{n\to\infty} N^{1/n}(nx)$ also satisfies (1) and (2). In addition, $N^*(nx) = (N^*(x))^n$.

We are interested in functions M on G such that

(i) $0 \leqq M \leqq N$,

(ii) $M(x+y) = M(x)M(y)$.

For such functions there is an analogue of the Hahn-Banach Theorem. Let $H \subseteq G$ be a subsemi-group of G and M a non-negative function on H satisfying (ii) for all $x, y \in H$ and

(iii) $M(x+h) \leqq M(h)N(x)$ whenever $x \in G$, $h \in H$, and $h+x \in H$. Such a function admits an extension $\overline{M}$ to all of G fulfilling (i) and (ii).

This extension criterion is suggested by the theorem of K. A. Ross [1], which is concerned with complex-valued functions ("semi-characters"). In a supplement to this note the present result will be combined with the Gelfand theory of commutative Banach algebras to give an extension of Ross' theorem.

Proof. The extension $\overline{M}$ will first be constructed under this additional hypothesis:

(A) For each $x \in G$, there exists $y \in G$ such that $x+y \in H$ and $M(x+y) > 0$.

It follows from (A) that $M > 0$ everywhere on H, for if $h \in H$, $y+h \in H$, $M(y+h) \leqq N(y)M(h)$. Similarly, if $h_i \in H$ and $x+h_1 = x+h_2$, $M(h_1) = M(h_2)$.

The problem can now be reduced to the case of a semi-group with cancellation. Specifically, let R be the equivalence relation in $G \times G$ of pairs (x, y) such that $x+z = y+z$ for some $z \in G$. Using associativity and commutativity it is easily verified that R is indeed an equivalence on G. More is true: if we set $\pi(x) = R[x]$, $\pi(G)$ has a unique semi-group structure such that π is a homomorphism of G onto $\pi(G)$. Also, $\pi(G)$ is a cancellation semi-group. Its elements are denoted by α, β, γ and $x \in \alpha$ means $\pi(x) = \alpha$.

Received by the editors July 17, 1964 and, in revised form, December 4, 1964.

[1] This work was done while the author was a pre-doctoral fellow of the National Science Foundation.

Define $L(\alpha)=\inf\{N(x);\ x\in\alpha\}$. From the usual argument for normed linear spaces it is clear that L fulfills (1) and (2) in place of N. If $\pi(h_1)=\pi(h_2)$, $h_i\in H$, $M(h_1)=M(h_2)$. There is thus a multiplicative functional E on $\pi(H)$, defined by $E\circ\pi=M$. (A) is still true.

To complete the reduction of the problem, it is necessary to prove the analogue of (iii),

(iii)′ $E(\alpha+\beta)\leqq E(\alpha)L(\beta)$, whenever $\alpha\in\pi(H)$, $\beta\in\pi(G)$, $\alpha+\beta\in\pi(H)$.

Let $h_1\in H\cap\alpha$, $x\in\beta$. $\pi(h_1+x)\in\pi(H)$ means the existence of $y\in G$ and $h_2\in H$ such that $h_1+x+y=h_2+y$. Using (A), we can replace y by $h_3\in H$: $h_1+x+h_3=h_2+h_3$. $M(h_2)M(h_3)\leqq M(h_3)M(h_1)N(x)$ by (i) and (iii). Since $M>0$ on H, $M(h_2)\leqq M(h_1)N(x)$. Thus $E(\alpha+\beta)=M(h_2)\leqq M(h_1)N(x)=E(\alpha)N(x)$. Since $x\in\beta$ was arbitrary, (iii)′ holds.

If an extension, say $\overline{E}$, is possible in this reduced case, $\overline{M}=\overline{E}\circ\pi$ is *a fortiori* an admissible extension for the given semi-group G. It is more convenient to return to that G and introduce the cancellation law as hypothesis for the remainder of the proof.

For $X\in G$ define $\mu=\mu(x)=\mathrm{Sup}\{M(y+nx+h)/N^*(y)M(h)\}^{1/n}$, the supremum being taken over all y, $n\geqq 1$, $h\in H$ such that $y+nx+h\in H$. It is easy to replace N by N^* in requirement (iii). Therefore

$$M(y+nx+h)\leqq N^*(y+nx)M(h)\leqq N^*(y)N^*(x)^nM(h).$$

Thus $\mu\leqq N^*(x)$.

If $\overline{M}$ were an extension to all of G satisfying (i) and (ii), necessarily $\overline{M}(x)\geqq\mu$. We shall now show that $\overline{M}(x)=\mu$ defines an extension of M to the subsemi-group H' determined by H and $\{x\}$ which satisfies (i)–(iii). This done, the existence of an extension to G follows from Zorn's Lemma.

Let $y+n_1x+h_1=n_2x+h_2$, $n_i\geqq 0$. Claim:

(iv) $N^*(y)\mu^{n_1}M(h_1)\geqq\mu^{n_2}M(h_2)$.

Because G has cancellation, we can take $n_1=0$ or $n_2=0$. If both are zero, (iv) follows from (iii). If $n_1\geqq 1$, $n_2=0$,

$$\mu^{n_1}\geqq M(y+n_1x+h_1)/N^*(y)M(h_1).$$

This is precisely (iv). The case $n_2>0$ follows:

Let $y_1+h_1=n_2x+h_2$ and $y_3+n_3x+h_3\in H$. In (iv) we replace μ by $\{M(y_3+n_3x+h_3)/N^*(y_3)M(h_3)\}^{1/n_3}$:

$$\begin{aligned}N^*(y_1)M(h_1)&\geqq\{M(y_3+n_3x+h_3)/N^*(y_3)M(h_3)\}^{n_2/n_3}M(h_2)\\&\Leftrightarrow N^*(n_3y_1)N^*(n_2y_3)M(n_3h_1+n_2h_3)\\&\geqq M(n_2y_3+n_2n_3x+n_2h_3+n_3h_2).\end{aligned}$$

But

$$n_2h_3 + n_2y_3 + n_2n_3x + n_3h_2 = n_2h_3 + n_2y_3 + n_3(y_1 + h_1)$$
$$= (n_2h_3 + n_3h_1) + (n_3y_1 + n_2y_3).$$

Hence

$$M(n_3h_2 + n_2y_3 + n_2n_3x + n_2h_3) \leqq M(n_2h_3 + n_3h_1)N^*(n_3y_1)N^*(n_2y_3).$$

Since y_3, n_3, h_3 were arbitrary, we can pass to the supremum to obtain μ in (iv).

Now suppose $n_1x+h_1=n_2x+h_2$, $n_i\geqq 0$. For every positive integer m,

$$m(n_1 + 1)x + mh_1 = m(n_2 + 1)x + mh_2.$$
$$x + (mn_1 + m - 1)x + mh_1 = m(n_2 + 1)x + mh_2.$$

If we apply (iv) to this, taking $x=y$, and let $m\to\infty$, we obtain $M(h_1)\mu^{n_1}\geqq M(h_2)\mu^{n_2}$. From this it follows that if we set $\overline{M}(x)=\mu$ on the subsemi-group $H'=\{H+nx;\ n\geqq 0\}\cup\{nx;\ n\geqq 1\}$ $\overline{M}$ is well-defined and multiplicative. The claim that $\overline{M}$ and H' satisfy (iii) is essentially contained in (iv).

To remove the restriction imposed by assuming (A), let $B=\{x;\ y+x\in H,\ M(y+x)>0,$ for some y in $G\}$. Almost by definition, (A) holds in the subsemi-group B. Let $\overline{M}_B$ be an extension of M, on $H\cap B$.

The complement CB is an ideal in G, that is, $G+CB\subseteq CB$. We define $\overline{M}\equiv 0$ on CB. Since $M=0$ on $H\cap CB$, $\overline{M}$ as defined now on all of G is the desired extension.

Reference

1. K. A. Ross, *A note on extending semicharacters to semigroups*, Proc. Amer. Math. Soc. 10 (1959), 579–583.

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