TWO ELEMENTARY THEOREMS ON THE INTERPOLATION OF LINEAR OPERATORS[1]

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Both theorems have to do with functions satisfying Hölder conditions.

DEFINITION. Let T be an operator which takes functions whose domain is n-space into functions whose domain is a metric space. T will be said to be of *Hölder type* (α, β) *norm* N if for $g = Tf$,

$$|f(x) - f(x-h)| \leqq A|h|^{\alpha} \quad \text{for all } x \text{ and } h,$$

implies

$$|g(u) - g(v)| \leqq NA|u-v|^{\beta} \quad \text{for all } u \text{ and } v.$$

(Throughout this paper, when dealing with a metric space we shall denote the distance between u and v by $|u-v|$.)

THEOREM 1. *Suppose that* $0 \leqq \alpha_0 \leqq \alpha_1 \leqq 1$, $\beta_0 \geqq 0$, $\beta_1 \geqq 0$ *and that* T *is a linear operator taking functions whose domain is n-space into functions whose domain is a metric space. If* T *is simultaneously of Hölder type* (α_0, β_0) *norm* N_0 *and of Hölder type* (α_1, β_1) *norm* N_1 *and if* $0 \leqq t \leqq 1$, *then* T *is of Hölder type* (α, β) *norm* N *where*

$$\alpha = \alpha_t = \alpha_0(1-t) + \alpha_1 t,$$
$$\beta = \beta_t = \beta_0(1-t) + \beta_1 t,$$
$$N \leqq R_n N_0^{1-t} N_1^{t},$$

and where R_n *depends only on the dimension of n-space.*

PROOF. Without loss of generality we may assume

$$|f(x) - f(x-h)| \leqq |h|^{\alpha}.$$

We first prove the theorem in case the domain of f is the real line, that is when $n = 1$.

For $r > 0$, let

$$K_r(s) = \begin{cases} 1/r - |s|/r^2 & \text{if } |s| < r, \\ 0 & \text{if } |s| \geqq r. \end{cases}$$

Then

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$$\int K_r(s)\,ds = \int_{|s|<r} K_r(s)\,ds = 1,$$

$$\int K_r'(s)\,ds = 0,$$

and,

$$\int |K_r'(s)|\,ds = 2/r.$$

Let

$$f_r(x) = \int f(x-s)K_r(s)\,ds = \int f(s)K_r(x-s)\,ds.$$

Let $\epsilon_r(x) = f(x) - f_r(x)$, $g = Tf$, $g_r = Tf_r$, and $\eta_r = T\epsilon_r$. Then $g = g_r + \eta_r$ by linearity of T.

$$f_r'(x) = \int f(s)K_r'(x-s)\,ds = \int f(x-s)K_r'(s)\,ds$$

$$= \int (f(x-s) - f(x))K_r'(s)\,ds.$$

$$|f_r'(x)| \leqq \int_{|s|<r} |f(x-s) - f(x)|\,|K_r'(s)|\,ds \leqq r^\alpha(2/r) = 2r^{\alpha-1}.$$

Case 1. $|h| < r$.

$$|f_r(x) - f_r(x-h)| \leqq |h| \sup_y |f_r'(y)| \leqq 2|h|r^{\alpha-1}$$

$$= 2|h|^{\alpha_1}|h|^{1-\alpha_1}r^{\alpha-1} \leqq 2|h|^{\alpha_1}r^{\alpha-\alpha_1}.$$

Case 2. $|h| \geqq r$.

$$|f_r(x) - f_r(x-h)| = \left|\int (f(x-s) - f(x-h-s))K_r(s)\,ds\right|$$

$$\leqq |h|^\alpha = |h|^{\alpha_1}|h|^{\alpha-\alpha_1} \leqq |h|^{\alpha_1}r^{\alpha-\alpha_1}.$$

In either case, f_r satisfies a Hölder condition of order α_1, indeed,

$$|f_r(x) - f_r(x-h)| \leqq 2r^{\alpha-\alpha_1}|h|^{\alpha_1}.$$

Thus,

$$|g_r(u) - g_r(v)| \leqq N_1 2r^{\alpha-\alpha_1}|u-v|^{\beta_1}.$$

$$\epsilon_r(x) = f(x) - f_r(x) = \int (f(x) - f(x-s))K_r(s)\,ds.$$

$$|\epsilon_r(x)| \leqq \int_{|s|<r} |s|^\alpha K_r(s)\,ds \leqq r^\alpha.$$

Case 1. $|h| \geqq r$.

$$|\epsilon_r(x) - \epsilon_r(x-h)| \leqq 2r^\alpha \leqq 2r^{\alpha-\alpha_0}|h|^{\alpha_0}.$$

Case 2. $|h| < r$.

$$\begin{aligned} |\epsilon_r(x) - \epsilon_r(x-h)| &\leqq |f(x) - f(x-h)| + |f_r(x) - f_r(x-h)| \\ &\leqq |h|^\alpha + |h|^\alpha \leqq 2|h|^{\alpha_0} r^{\alpha-\alpha_0}. \end{aligned}$$

Thus ϵ_r satisfies a Hölder condition of order α_0. Therefore,

$$|\eta_r(u) - \eta_r(v)| \leqq N_0 2r^{\alpha-\alpha_0}|u-v|^{\beta_0}.$$

Thus, if we set $r = (N_1|u-v|^{\beta_1-\beta_0}/N_0)^{1/(\alpha_1-\alpha_0)}$,

$$\begin{aligned} |g(u) - g(v)| &\leqq |g_r(u) - g_r(v)| + |\eta_r(u) - \eta_r(v)| \\ &\leqq 2N_1 r^{\alpha-\alpha_1}|u-v|^{\beta_1} + 2N_0 r^{\alpha-\alpha_0}|u-v|^{\beta_0} \\ &= 4N_0^{1-t}N_1^t|u-v|^\beta. \end{aligned}$$

This proves the theorem when the domain of f is one dimensional. For $n>1$, the case $n=2$ is already sufficiently general to illustrate the proof. In this case we let

$$K_r(s) = \begin{cases} 3/\pi r^2 - 3|s|/\pi r^3 & \text{if } |s| < r, \\ 0 & \text{if } |s| \geqq r. \end{cases}$$

For a given $h=(h_1, h_2)\neq 0$, let $\partial/\partial\theta$ denote directional differentiation in the direction $\theta = h/|h|$. Then $\int(\partial/\partial\theta)K_r(s)\,ds$ vanishes and $\int|(\partial/\partial\theta)K_r(s)|\,ds = O(1/r)$. Thus,

$$\begin{aligned} |(\partial/\partial\theta)f_r(x)| &\leqq \int_{|s|<r} |f(x-s) - f(x)|\,|(\partial/\partial\theta)K_r(s)|\,ds \\ &\leqq r^\alpha O(1/r) = O(r^{\alpha-1}). \end{aligned}$$

Therefore, if $0<|h|<r$, $\theta = h/|h|$,

$$\begin{aligned} |f_r(x) - f_r(x-h)| &\leqq |h| \sup_y |(\partial/\partial\theta)f_r(y)| = |h|\,O(r^{\alpha-1}) \\ &= O(|h|^{\alpha_1} r^{\alpha-\alpha_1}). \end{aligned}$$

The rest of the proof goes through as before.

DEFINITION. An operator T is said to *take* L^p *into* Lip α *with norm* N if for $g=Tf$,

$$|g(u) - g(v)| \leqq N\|f\|_p |u - v|^\alpha \quad \text{for all } u \text{ and } v.$$

If f is a measurable function and $y>0$, let

$$m(f, y) = m(|f|, y) = \text{measure of } \{x: |f(x)| > y\}.$$

It is easily shown that

$$\int |f(x)|\, dx = \int_0^\infty m(f, y)\, dy.$$

Furthermore, for $p>0$,

$$\begin{aligned} m(|f|^p, y) &= \text{meas}\{x: |f(x)|^p > y\} \\ &= \text{meas}\{x: |f(x)| > y^{1/p}\} = m(f, y^{1/p}). \end{aligned}$$

Thus,

$$\begin{aligned} (\|f\|_p)^p &= \int |f(x)|^p\, dx = \int_0^\infty m(|f|^p, v)\, dv \\ &= \int_0^\infty m(f, v^{1/p})\, dv = p \int_0^\infty m(f, y) y^{p-1}\, dy. \end{aligned}$$

Given $k \geqq 0$, let

$$f_k(x) = \begin{cases} f(x) & \text{if } |f(x)| \leqq k, \\ k \operatorname{sgn} f(x) & \text{if } |f(x)| > k, \end{cases}$$

and let

$$f^k(x) = f(x) - f_k(x).$$

THEOREM 2. *Suppose that* $0<p_0\leqq p_1\leqq\infty$, $\alpha_0\geqq 0$, $\alpha_1\geqq 0$, *and that* T *is a linear operator taking measurable functions on a measure space into functions whose domain is a metric space. If* T *simultaneously takes* L^{p_0} *into* Lip α_0 *with norm* N_0 *and* L^{p_1} *into* Lip α_1 *with norm* N_1 *and if* $0\leqq t\leqq 1$, *then* T *takes* L^p *into* Lip α *with norm* N *where*

$$\begin{aligned} 1/p &= 1/p_t = (1 - t)/p_0 + t/p_1, \\ \alpha &= \alpha_t = (1 - t)\alpha_0 + t\alpha_1, \\ N &\leqq N_0^{1-t} N_1^t/(1 - t)^{1-t} t^t \leqq 2N_0^{1-t} N_1^t. \end{aligned}$$

(*It is to be remarked that* $1/(1-t)^{1-t}t^t$ *tends to* 1 *as* t *tends to* 0 *or* 1.)

PROOF. Suppose, without loss of generality, that $||f||_p=1$. Fix $k\geqq 0$, then $f=f^k+f_k$. Let $g=Tf$, $g_0=Tf^k$ and $g_1=Tf_k$, then $g=g_0+g_1$ by linearity of T.

$$
\begin{aligned}
(||f^k||_{p_0})^{p_0} &= p_0\int_0^\infty y^{p_0-1}m(f^k, y)\,dy = p_0\int_0^\infty y^{p_0-1}m(f, y+k)\,dy\\
&= p_0\int_k^\infty (z-k)^{p_0-1}m(f, z)\,dz \leqq p_0\int_k^\infty z^{p_0-1}m(f, z)\,dz\\
&\leqq p_0k^{p_0-p}\int_k^\infty z^{p-1}m(f, z)\,dz \leqq (p_0k^{p_0-p}/p)(||f||_p)^p\\
&= p_0k^{p_0-p}/p.
\end{aligned}
$$

Thus $||f^k||_{p_0}\leqq (p_0/p)^{1/p_0}k^{1-p/p_0}$; since T takes L^{p_0} into Lip α_0 with norm N_0,

$$
|g_0(u)-g_0(v)| \leqq N_0(p_0/p)^{1/p_0}k^{1-p/p_0}|u-v|^{\alpha_0}.
$$

$$
\begin{aligned}
(||f_k||_{p_1})^{p_1} &= p_1\int_0^\infty y^{p_1-1}m(f_k, y)\,dy = p_1\int_0^k y^{p_1-1}m(f, y)\,dy\\
&\leqq p_1k^{p_1-p}\int_0^k y^{p-1}m(f, y)\,dy \leqq p_1k^{p_1-p}/p.
\end{aligned}
$$

Thus $||f_k||_{p_1}\leqq (p_1/p)^{1/p_1}k^{1-p/p_1}$, and this last equation is valid even if $p_1=\infty$.

$$
|g_1(u)-g_1(v)| \leqq N_1(p_1/p)^{1/p_1}k^{1-p/p_1}|u-v|^{\alpha_1}.
$$

If we set $A = 1/p_0 - 1/p_1$, then $1/p - 1/p_1 = A(1-t)$ and $1/p_0-1/p=At$.

Thus, if we let

$$
k^{pA} = (t/1-t)(p_0/p)^{1/p_0}(p_1/p)^{-1/p_1}(N_0/N_1)|u-v|^{\alpha_0-\alpha_1},
$$

$$
\begin{aligned}
|g(u)-g(v)| &\leqq |g_0(u)-g_0(v)| + |g_1(u)-g_1(v)|\\
&= N_0(p_1/p)^{1/p_0}k^{-pAt}|u-v|^{\alpha_0}\\
&\quad + N_1(p_1/p)^{1/p_1}k^{pA(1-t)}|u-v|^{\alpha_1}\\
&= N_0^{1-t}N_1^t(1/t^t(1-t)^{1-t})(p_0/p)^{1-t/p_0}(p_1/p)^{t/p_1}|u-v|^{\alpha}.
\end{aligned}
$$

Let

$$
B = (p_0/p)^{(1-t)/p_0}(p_1/p)^{t/p_1},
$$

then

$\log B = (1/p)\log(1/p) - (1-t)(1/p_0)\log(1/p_0) - t(1/p_1)\log(1/p_1).$

But $x \log x$ is a convex function of $x \geqq 0$, so that

$$(1/p)\log(1/p) \leqq (1-t)(1/p_0)\log(1/p_0) + t(1/p_1)\log(1/p_1).$$

Thus $\log B \leqq 0$, $B \leqq 1$ and the theorem is established.

REMARK. It is possible to strengthen the result of Theorem 2. We shall say that a measurable function f belongs to *weak* L^p if there exists a number A such that for all $y > 0$,

$$m(f, y) \leqq (A/y)^p.$$

If $f \in L^p$ then f belongs to weak L^p, since

$$(\|f\|_p)^p = p\int_0^\infty m(f, u)u^{p-1}\,du \geqq p\int_0^y m(f, u)u^{p-1}\,du$$

$$\geqq pm(f, y)\int_0^y u^{p-1}\,du = m(f, y)y^p.$$

Thus,

$$m(f, y) \leqq (\|f\|_p/y)^p.$$

We shall say that a function $f \in \operatorname{Lip} \alpha$ if for all u and v,

$$|f(u) - f(v)| \leqq A|u - v|^\alpha.$$

We shall say $f \in \operatorname{Lip} \alpha$ if

$$|f(u) - f(v)| = o(|u - v|^\alpha)$$

as $|u - v|$ tends to zero or infinity.

1°. Under the hypotheses of Theorem 2, T takes weak L^p into $\operatorname{Lip} \alpha$ if $p_0 < p < p_1$.

2°. Under the hypotheses of Theorem 2, T takes L^p into $\operatorname{Lip} \alpha$ if $p_0 < p < p_1$.

To prove 1°, we suppose that $m(f, y) \leqq 1/y^p$. Then

$$(\|f^k\|_{p_0})^{p_0} \leqq p_0\int_k^\infty z^{p_0-1}m(f, z)\,dz \leqq (p_0/p - p_0)k^{p_0-p},$$

$$\|f^k\|_{p_0} \leqq (p_0/p - p_0)^{1/p_0}k^{1-p/p_0}.$$

Similarly,

$$\|f_k\|_{p_1} \leqq (p_1/p_1 - p)^{1/p_1}k^{1-p/p_1}.$$

Thus, if we let

$$k^{pA} = (N_0/N_1)\,|\,u - v\,|^{\alpha_0-\alpha_1},$$

$$|\,g(u) - g(v)\,| \leqq N_0^{1-t}N_1^{t}\,|\,u - v\,|^{\alpha}\left\{(p_0/p - p_0)^{1/p_0} + (p_1/p_1 - p)^{1/p_1}\right\}.$$

To prove 2°, we observe that

$$(\|f\|_p)^p = \int_0^\infty m(f,\, v^{1/p})\, dv.$$

Since $m(f,\, v^{1/p})$ is a monotone function of v, the finiteness of the integral implies

$$m(f,\, v^{1/p}) = o(1/v) \text{ as } v \text{ tends to zero or infinity.}$$

Thus,

$$m(f,\, y) = o(1/y^p) \text{ as } y \text{ tends to zero or infinity.}$$

Therefore,

$$\|f^k\|_{p_0} = o(k^{1-p/p_0}) \text{ as } k \text{ tends to zero or infinity,}$$

and

$$\|f_k\|_{p_1} = o(k^{1-p/p_1}) \text{ as } k \text{ tends to zero or infinity.}$$

Again we may let

$$k^{pA} = (N_0/N_1)\,|\,u - v\,|^{\alpha_0-\alpha_1}.$$

Thus,

$$|\,g(u) - g(v)\,| = o(|\,u - v\,|^{\alpha}).$$

References

1. R. Fiorenza, *Theoremi di interpolazione per transformazioni tra spazi di funzioni con derivate hölderiane*, Ricerche Mat. **10** (1961), 282–298.

2. ———, *Ulteriori risultati sulla interpolazione tra spazi di funzioni con derivate hölderiane*, Richerche Mat. **11** (1962), 224–253.

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