ISOMORPHISM TYPES OF INDEX SETS OF PARTIAL RECURSIVE FUNCTIONS

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1. Let $\{q_0, q_1, q_2, \cdots\}$ be a Kleene enumeration of partial recursive functions. If f is such a function, denote by θf its *index set*, $\theta f = \{n \mid q_n \simeq f\}$. Insofar as the indices of a partial recursive function correspond to the different sets of "instructions" for computing its values, it is natural to ask how much of the "complexity" of the function is reflected by its index set; for example, one might expect the index set of a constant total function to differ in a fundamental way from that of a function whose domain and range are nonrecursive sets. More precisely, since the basic equivalence relation of recursion theory is *recursive isomorphism* (i.e., equivalence under a recursive permutation of the nonnegative integers) the following question arises: How many distinct recursive isomorphism types of index sets are there, and which properties of the corresponding functions can be used to characterize these types? This is answered in the theorem below. That the answer is independent of any particular enumeration follows from the fact, proved by Rogers in [**5**], that different "standard-type" enumerations are related by means of recursive permutations.

In the following, *function* will mean *partial recursive function* and *degree* will mean *Turing degree of unsolvability*. We write $\alpha R_1 \beta$ for α *is* 1-1 *reducible to* β and $\alpha \cong \beta$ for α *is recursively isomorphic to* β. The notation is that of [**3**], but the technique will be informal in character.

THEOREM 1. *There are exactly three isomorphism types of index sets of partial recursive functions, and the type of θf is uniquely determined by whether the domain of f is null, finite or infinite. In the first two cases, θf has degree $\mathbf{0}'$, in the last case degree $\mathbf{0}''$.*

The proof follows from several lemmas.

LEMMA 1. *Let f, g be finite, nonvoid functions. Then $\theta f \cong \theta g$.*

PROOF. Assume $f = \{(a_0, b_0), \cdots, (a_p, b_p)\}$ and $g = \{(x_0, y_0), \cdots, (x_m, y_m)\}$. Let x_{m+1} be distinct from $x_0, \cdots, x_m$. Using standard techniques, define a 1-1 recursive function h by

$$q_{h(n)}(x_i) = y_i \quad \text{for } 0 \leqq i \leqq m \text{ if } q_n(a_j) = b_j \text{ for } 0 \leqq j \leqq p,$$

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$$q_{h(n)}(x_{m+1}) = 0 \quad \text{if } q_n(a_j) = b_j \text{ for } 0 \leqq j \leqq p \textit{ and } q_n(x) \text{ is defined for some } x \notin \{a_0, \cdots, a_p\},$$

$$q_{h(n)}(x) \text{ undefined otherwise.}$$

(We note here for future reference that for all n, either $q_{h(n)} = \varnothing \subseteq g$ or $q_{h(n)}$ extends g.)

Then

$$h(n) \in \theta g \leftrightarrow q_{h(n)}(x_i) = y_i \text{ for } 0 \leqq i \leqq m \text{ and } q_{h(n)}(x) \text{ is undefined for all } x \notin \{x_0, \cdots, x_m\}$$
$$\leftrightarrow q_n(a_j) = b_j \text{ for } 0 \leqq j \leqq p \text{ and } q_n(x) \text{ is undefined for all } x \notin \{a_0, \cdots, a_p\}$$
$$\leftrightarrow n \in \theta f.$$

So θf is 1-1 reducible to θg. By symmetry, $\theta g R_1 \theta f$, which by Myhill's theorem [**4**] implies $\theta f \cong \theta g$.

LEMMA 2. *Assume q_e has an infinite domain, and let α be any set whose defining predicate can be written in **AE** form in the arithmetic hierarchy. Then α is* 1-1 *reducible to θq_e.*

PROOF. This was shown for the case where q_e is total by Shapiro in [**6**]. His argument easily generalizes to functions with arbitrary infinite domains, as follows: Assume that $n \in \alpha \equiv (z)(Ey)R(n, z, y)$ for a recursive R. Now define a recursive function d by

$$q_{d(n)}(z) = \text{the } z\text{th number } x \text{ for which } q_n(x) \text{ is computed in some simultaneous computation of all values of } q_n.$$

Then $q_{d(e)}$ is clearly total if domain q_e is infinite, and range $q_{d(e)}$ = domain q_e. Now define a 1-1 recursive function h by

$$q_{h(n)}(x) = q_e(x) \quad \text{if } (Ey)(Ez)(x = q_{d(e)}(z) \wedge R(n, z, y)),$$
$$q_{h(n)}(x) \quad \text{undefined otherwise.}$$

(We again note, for future reference, that q_e extends $q_{h(n)}$ for each n.)

Then

$$n \in \alpha \leftrightarrow (z)(Ey)R(n, z, y) \leftrightarrow \text{domain } q_{h(n)} \subseteq \text{domain } q_e \text{ and}$$
$$(x)[x \in \text{domain } q_e \supset (Ez)(x = q_{d(e)}(z) \wedge (Ey)R(n, z, y))]$$
$$\leftrightarrow q_{h(n)} \simeq q_e \leftrightarrow h(n) \in \theta q_e. \text{ So } \alpha R_1 \theta q_e.$$

It is evident that a Gödel number of h can be computed, uniformly in e and a Gödel number of R, although we shall not use this fact.

LEMMA 3. *For any e, the defining predicate of θq_e can be written in* **AE** *form in the arithmetic hierarchy.*

PROOF.

$$\begin{aligned} n \in \theta q_e \leftrightarrow\ & (z)[(Ey)T_1(e, z, y) \wedge (Ey)T_1(n, z, y) \wedge U(\mu y T_1(e, z, y)) \\ & = U(\mu y T_1(n, z, y)) . \vee . (y)(\neg T_1(e, z, y) \wedge \neg T_1(n, z, y)] \\ \leftrightarrow\ & (z)(Ey)[T_1(e, (z)_1, (y)_1) \wedge T_1(n, (z)_1, (y)_2) \wedge U((y)_1) \\ & = U((y)_2) . \vee . \neg T_1(e, (z)_1, (z)_2) \wedge \neg T_1(n, (z)_1, (z)_2)] \end{aligned}$$

where the scope of the quantifiers is a recursive predicate.

PROOF OF THEOREM. Let $\emptyset$ denote the null function, and let T_0 be the isomorphism type containing $\theta\emptyset$. Let T_1 be the isomorphism type of θf where f is any finite nonvoid function; this is well defined by Lemma 1. That $T_0 \neq T_1$ follows from the fact that $(\theta\emptyset)'$ is recursively enumerable while, as shown in [**1**], $(\theta f)'$ is productive for any $f \neq \emptyset$. Since $(\theta\emptyset)'$ is creative, it follows from [**4**] that degree $(\theta\emptyset) =$ degree $(\theta\emptyset)' = \mathbf{0}'$. By an argument of [**1**] (given there for sets rather than functions), if f is any finite function, degree $\theta f = \mathbf{0}'$. Now if f and g are any two functions with infinite domains, it follows from Lemmas 2 and 3 that $\theta f R_1 \theta g$ and $\theta g R_1 \theta f$, so that $\theta f \cong \theta g$. It also follows from those lemmas that these sets are of degree $\mathbf{0}''$, so that if T_2 denotes their isomorphism type, T_2 is distinct from both T_0 and T_1.

It may further be noted that if, for a recursively enumerable (r.e.) set α, $\theta\alpha$ denotes $\{n \mid \alpha = \omega_n = \text{range } q_n\}$, then the reduction procedures of Lemmas 1 and 2 can (with minor modifications) be used to show that the isomorphism type of $\theta\alpha$ is uniquely determined by whether α is null, finite or infinite, and that in fact that three possible types of $\theta\alpha$ are exactly those of Theorem 1. In particular, all infinite r.e. sets, independently of their degree of unsolvability, have recursively isomorphic index sets.

2. In [2], it is shown that if f, g are partial recursive functions neither of which extends the other, the pair of index sets $(\theta f, \theta g)$ possesses the property of inducing double creativity of any pair (α, β) of disjoint r.e. sets such that $\theta f \subseteq \alpha$, $\theta g \subseteq \beta$. It is then relevant to inquire how many such pairs of index sets there are, up to double isomorphism (i.e., simultaneous isomorphism under a single recursive permutation). This is answered by the theorem below.

DEFINITION 1. A partial recursive function has *type* 0, 1 or 2 according as its domain is null, finite or infinite. The type of f will be denoted by $T(f)$.

DEFINITION 2. Functions f, g will be called *comparable* if one extends the other; otherwise, f and g are *incomparable.*

PROPOSITION 1. *For any pair f, g of partial recursive functions, there exist disjoint r.e. sets α and β with $\theta f \subseteq \alpha$, $\theta g \subseteq \beta$ if and only if f and g are incomparable.*

PROOF. This is a special case of Theorems 4.1 and 4.2 of [1].

THEOREM 2. *Let f_0, f_1 be incomparable functions. Then $(\theta f_0, \theta f_1)$ is doubly isomorphic to $(\theta g_0, \theta g_1)$ if and only if the following conditions hold:*

(1) $T(f_0) = T(g_0)$ and $T(f_1) = T(g_1)$,

(2) g_0 and g_1 are incomparable.

PROOF. By Proposition 1, the incomparability of f_0 and f_1 implies the existence of disjoint r.e. sets β_0, β_1 with $\theta f_0 \subseteq \beta_0$, $\theta f_1 \subseteq \beta_1$. To prove the necessity of the above conditions, assume that π is a recursive permutation such that $\pi(\theta f_0) = \theta g_0$ and $\pi(\theta f_1) = \theta g_1$. Condition (1) then follows from Theorem 1. Moreover, $\theta g_0 = \pi(\theta f_0) \subseteq \pi(\beta_0)$ and $\theta g_1 = \pi(\theta f_1) \subseteq \pi(\beta_1)$ where $\pi(\beta_0)$, $\pi(\beta_1)$ are disjoint r.e. sets, which by Proposition 1 implies condition (2). To prove sufficiency, assume that g_0, g_1 are incomparable, $T(f_0) = T(g_0)$, $T(f_1) = T(g_1)$. For $i = 0, 1$ let h_i be the 1-1 recursive function which reduces θf_i to θg_i provided by the lemmas to Theorem 1; recall that, as noted above, these reductions have the property that for each n, $q_{h_i(n)}$ is comparable to g_i. Now define a 1-1 recursive function h by

$$q_{h(n)}(x) \simeq q_{h_0(n)}(x) \quad \text{if } n \in \beta_0,$$

$$q_{h(n)}(x) \simeq q_{h_1(n)}(x) \quad \text{if } n \in \beta_1,$$

$$q_{h(n)}(x) \text{ undefined otherwise.}$$

We note first that for $i = 0, 1$ we have

$$n \in \theta f_i \to n \in \beta_i \to q_{h(n)} \simeq q_{h_i(n)},$$

so that $h(n) \in \theta g_i \leftrightarrow h_i(n) \in \theta g_i$. But by choice of h_i, $n \in \theta f_i \leftrightarrow h_i(n) \in \theta g_i$. Thus $n \in \theta f_i \to h(n) \in \theta g_i$.

Conversely, assume $h(n) \in \theta g_i$, i.e., $q_{h(n)} \simeq g_i$. We then observe the following:

(a) $n \in (\beta_0 \cup \beta_1)' \to q_{h(n)} = \varnothing \to g_i = \varnothing$, which contradicts the hypothesis that g_i and g_{1-i} are incomparable.

(b) $n \in \beta_{1-i} \to q_{h_{1-i}(n)} \simeq q_{h(n)} \simeq g_i$. But as noted above, $q_{h_{1-i}(n)}$ and thus g_i is comparable to g_{1-i}, which again contradicts the hypothesis.

We thus deduce that $h(n) \in \theta g_i \to n \in \beta_i$. But $n \in \beta_i \to q_{h(n)} \simeq q_{h_i(n)}$, while $q_{h_i(n)} \simeq g_i \leftrightarrow n \in \theta f_i$. Together, these yield $h(n) \in \theta g_i \to n \in \theta f_i$.

We have thus shown that $\theta f_0 = h^{-1}(\theta g_0)$ and $\theta f_1 = h^{-1}(\theta g_1)$, i.e., that h is a 1-1 reduction of the pair $(\theta f_0, \theta f_1)$ to $(\theta g_0, \theta g_1)$. By symmetry of the hypotheses, $(\theta g_0, \theta g_1)$ is 1-1 reducible to $(\theta f_0, \theta f_1)$. So by Smullyan's generalization [7] of Myhill's theorem, $(\theta f_0, \theta f_1)$ is doubly isomorphic to $(\theta g_0, \theta g_1)$.

There are thus exactly three isomorphism types of pairs of index sets of incomparable functions, corresponding to pairs of functions of types (1, 1), (1, 2) and (2, 2). The situation for pairs of comparable functions appears to be more complicated. We do not know conditions which are both necessary and sufficient for double isomorphism of such pairs of index sets.

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