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NOTE ON ANALYTICALLY UNRAMIFIED SEMI-LOCAL RINGS

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All rings in this paper are assumed to be commutative rings with a unit element. If B is an ideal in a ring R, the *integral closure* B_a of B is the set of elements x in R such that x satisfies an equation of the form $x^n+b_1x^{n-1}+ \cdots +b_n=0$, where $b_i\in B^i$ $(i=1, \cdots, n)$. An ideal B in R is *semi-prime* in case B is an intersection of prime ideals. If R is an integral domain, then R is *normal* in case R is integrally closed in its quotient field. If R is a semi-local (Noetherian) ring, then R is *analytically unramified* in case the completion of R (with respect to the powers of the Jacobson radical of R) contains no nonzero nilpotent elements.

Let R be a semi-local ring with Jacobson radical J, and let R^* be the completion of R. In [2], Zariski proved that if R is a normal local integral domain, and if there is a nonzero element x in J such that $\mathfrak{p}R^*$ is semi-prime, for every prime divisor $\mathfrak{p}$ of xR, then R is analytically unramified. In [1, p. 132] Nagata proved that if R is a semi-local integral domain, and if there is a nonzero element x in J such that, for every prime divisor $\mathfrak{p}$ of xR, $\mathfrak{p}R^*$ is semi-prime and $R_\mathfrak{p}$ is a valuation ring, then R is analytically unramified. (The condition $R_\mathfrak{p}$ is a valuation ring holds if R is normal.) The main purpose of this note is to extend Nagata's result to the case where R is a semi-local ring (Theorem 1). This extension will be given after first proving a

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number of lemmas. Among these preliminary results, Lemma 3 gives a necessary and sufficient condition for $R_{\mathfrak{p}}$ to be a discrete Archimedian valuation ring (where R is a Noetherian ring and $\mathfrak{p}$ is a prime divisor of a nonzero-divisor $b \in R$), Corollary 2 of Lemma 5 gives a sufficient condition for a Noetherian ring to be a direct sum of normal Noetherian domains, and Lemma 6 gives a characterization of analytically unramified semi-local rings.

In Lemmas 1–4 below, R is a Noetherian ring, S is the integral closure of R in its total quotient ring, b is a nonunit in R which is not a divisor of zero, $\mathfrak{p}$ is a prime divisor of bR, and $\mathfrak{q}$ is the isolated component of zero determined by $\mathfrak{p}$. If B is an ideal in R, then $B'R_{\mathfrak{p}}$ is the ideal generated by $(B+\mathfrak{q})/\mathfrak{q}$ in $R_{\mathfrak{p}}$. Likewise, if $c \in R$, then c' is the $\mathfrak{q}$-residue of c.

LEMMA 1. *$(bR)_a = bS \cap R$, and an element c in R is in $(bR)_a$ if and only if $c/b \in S$.*

PROOF. If $c \in (bR)_a$, then $c^n + b_1c^{n-1} + \cdots + b_n = 0$, where $b_i \in b^iR$. Dividing this equation by b^n shows that $c/b \in S$, so $c \in bS \cap R$, hence $(bR)_a \subseteq bS \cap R$. If $c \in bS \cap R$, then $c/b \in S$, so $(c/b)^n + r_1(c/b)^{n-1} + \cdots + r_n = 0$, where $r_i \in R$. Multiplying this equation by b^n shows that $c \in (bR)_a$, since $c \in R$. Therefore $bS \cap R \subseteq (bR)_a$, hence $(bR)_a = bS \cap R$, q.e.d.

LEMMA 2. *$R_{\mathfrak{p}}$ is a discrete Archimedian valuation ring if and only if $R_{\mathfrak{p}}$ is normal.*

PROOF. If $R_{\mathfrak{p}}$ is a valuation ring, then $R_{\mathfrak{p}}$ is normal. Conversely, if $R_{\mathfrak{p}}$ is normal, then $R_{\mathfrak{p}}$ is a normal local integral domain (hence, the kernel of the natural homomorphism from R into $R_{\mathfrak{p}}$, which is $\mathfrak{q}$, is a prime ideal), and $\mathfrak{p}'R_{\mathfrak{p}}$ is a prime divisor of $b'R_{\mathfrak{p}}$. Since $b'R_{\mathfrak{p}} \neq (0)$, height $\mathfrak{p}'R_{\mathfrak{p}} = 1$ hence $R_{\mathfrak{p}}$ is a discrete Archimedian valuation ring [**3**, pp. 276–278], q.e.d.

An element $c \in R$ such that $bR: cR = \mathfrak{p}$ is used in the next lemma. Such an element can be found as follows. Let $\mathfrak{p} = \mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_n$ be the prime divisors of bR, and let d be an element in the $\mathfrak{p}_i$-primary component of bR $(i = 2, \cdots, n)$ which is not in bR. If $bR: dR \neq \mathfrak{p}$, let e be an element in $(bR:dR):\mathfrak{p}R$ which is not in $bR:dR$, and let $c = de$.

LEMMA 3. *Let c be an element in R such that $bR:cR = \mathfrak{p}$. $R_{\mathfrak{p}}$ is normal if and only if $c/b \notin S$.*

PROOF. Let $R_{\mathfrak{p}}$ be normal. Since $bR:cR = \mathfrak{p}$, $b'R_{\mathfrak{p}}:c'R_{\mathfrak{p}} = \mathfrak{p}'R_{\mathfrak{p}}$. Therefore $c' \notin b'R_{\mathfrak{p}}$, so $c'/b' \notin R_{\mathfrak{p}}$. Hence, since $S_{R\sim\mathfrak{p}}$ is contained in the integral closure of $R_{\mathfrak{p}}$ in its quotient field, $c/b \notin S$. Conversely, assume $c/b \notin S$. Since $c\mathfrak{p} \subseteq bR$, $(c/b)\mathfrak{p} \subseteq R$. If $(c/b)\mathfrak{p} \subseteq \mathfrak{p}$, then $bR[c/b] \subseteq \mathfrak{p}R[c/b] \subseteq R$, so $R[c/b]$ is contained in the finite R-module $(1/b)R$, hence $c/b \in S$. This is a contradiction, so $c\mathfrak{p} \not\subseteq b\mathfrak{p}$. Therefore, there are elements $d \in \mathfrak{p}$, and $x \in R$, $\notin \mathfrak{p}$, such that $cd = bx$. Then $b'R_{\mathfrak{p}} = b'x'R_{\mathfrak{p}} = c'd'R_{\mathfrak{p}} \subseteq c'\mathfrak{p}'R_{\mathfrak{p}} \subseteq b'R_{\mathfrak{p}}$, so $c'\mathfrak{p}'R_{\mathfrak{p}} = b'R_{\mathfrak{p}} = c'd'R_{\mathfrak{p}}$. Now c' is not a divisor of zero in $R_{\mathfrak{p}}$ (since $b'x'$ is not), so $\mathfrak{p}'R_{\mathfrak{p}} = (b'/c')R_{\mathfrak{p}}$, hence $R_{\mathfrak{p}}$ is normal (Lemma 2, and [**3**, p. 277]), q.e.d.

LEMMA 4. *$(bR)_a = bR$ if and only if $R_{\mathfrak{p}}$ is normal, for every prime divisor $\mathfrak{p}$ of bR.*

PROOF. If $R_{\mathfrak{p}}$ is normal, for every prime divisor $\mathfrak{p}$ of bR, then $R_{\mathfrak{p}} = S_{R\sim\mathfrak{p}}$, so $\mathfrak{p}'R_{\mathfrak{p}} \cap S$ is a prime divisor of bS. Let $\mathfrak{p}_1, \cdots, \mathfrak{p}_n$ be the prime divisors of bR, and let b_i be the image of b in $R_{\mathfrak{p}_i}$. Then $(bR)_a = bS \cap R$ (Lemma 1) $\subseteq (\bigcap_1^n (b_iR_{\mathfrak{p}_i} \cap S)) \cap R = \bigcap_1^n (b_iR_{\mathfrak{p}_i} \cap R) = bR \subseteq (bR)_a$, hence $(bR)_a = bR$. Conversely, let $(bR)_a = bR$, let $\mathfrak{p}$ be a prime divisor of bR, and let c be an element in R such that $bR : cR = \mathfrak{p}$. Then $c/b \notin R$. If $R_{\mathfrak{p}}$ is not normal, then $c/b \in S$ (Lemma 3), hence $c \in bS \cap R = (bR)_a$ (Lemma 1). Since $(bR)_a = bR$, this is a contradiction to $c/b \notin R$. Therefore $R_{\mathfrak{p}}$ is normal, q.e.d.

LEMMA 5. *Let R be a Noetherian ring with Jacobson radical J, let b be a nonzero element in J, and let $\mathfrak{p}_1, \cdots, \mathfrak{p}_n$ be the prime divisors of bR. If $R_{\mathfrak{p}_i}$ is a discrete Archimedian valuation ring ($i = 1, \cdots, n$), then the isolated component of zero contained in $\mathfrak{p}_i$ is a prime ideal $\mathfrak{q}_i$ and $\bigcap_1^n \mathfrak{q}_i = (0)$. Moreover, b is not a zero-divisor in R, and $(bR)_a = bR$.*

PROOF. If $R_{\mathfrak{p}_i}$ is a discrete Archimedian valuation ring, then $R_{\mathfrak{p}_i}$ is an integral domain which is not a field, so the isolated component of zero contained in $\mathfrak{p}_i$ is a prime ideal $\mathfrak{q}_i$. Since $\mathfrak{q}_i$ is the kernel of the natural homomorphism from R into $R_{\mathfrak{p}_i}$, $\mathfrak{q}_i$ is contained in every $\mathfrak{p}_i$-primary ideal. Hence, since $bR = \bigcap_1^n (b_iR_{\mathfrak{p}_i} \cap R)$, where b_i is the $\mathfrak{q}_i$-residue of b, and since each $\mathfrak{p}_i$ is a minimal prime divisor of bR, $Z = \bigcap_1^n \mathfrak{q}_i \subseteq bR$. Since $b \notin \mathfrak{q}_i$ ($i = 1, \cdots, n$), $Z : bR = Z$. This implies $Z = bR \cap (Z : bR) = b(Z : bR)$. Therefore, since $b \in J$, $Z = b(Z : bR) = bZ \subseteq JZ \subseteq Z$. Hence, $Z = \bigcap_1^\infty J^h Z \subseteq \bigcap_1^\infty J^h = (0)$. Thus b is not a zero-divisor, so $(bR)_a = bR$ (Lemma 4), q.e.d.

COROLLARY 1. *With the same R and J of Lemma 5, suppose there is a nonzero nilpotent element in R. If b is a nonzero divisor in J, then $(bR)_a \neq bR$.*

PROOF. If b is a nonzero-divisor in J such that $(bR)_a = bR$, then $R_{\mathfrak{p}}$ is a discrete Archimedian valuation ring, for every prime divisor $\mathfrak{p}$ of bR (Lemma 4). Hence by Lemma 5, the zero ideal in R is semi-prime, q.e.d.

Corollary 4 below is the next result which is needed to prove Theorem 1, and it can be proved as a corollary to Lemma 5. Corollaries 1, 2, and 3, and Lemma 6 are not used in the proof of Theorem 1. They are included at this point because they are of some interest in themselves.

COROLLARY 2. *Let R be an integrally closed Noetherian ring, let J be the Jacobson radical of R, and let $\mathfrak{q}_1, \cdots, \mathfrak{q}_n$ be the minimal prime divisors of zero. If there is a nonzero-divisor b in J, then $R = \bigoplus_1^n R/\mathfrak{q}_i$, and $R/\mathfrak{q}_i$ is a normal Noetherian domain.*

PROOF. If b is a nonzero-divisor in J, then $(bR)_a = bR$, since R is integrally closed. Therefore by Corollary 1 the zero ideal in R is semi-prime, and consequently the total quotient ring Q of R is the direct sum of n fields. Since the idempotents in Q are integrally dependent on R, they are in R. This, and the fact that R is integrally closed, immediately imply the conclusions, q.e.d.

In Corollaries 3 and 4 and Lemmas 6 and 7, R is a semi-local ring with maximal ideals $M_1, \cdots, M_d$, $J = \bigcap_1^d M_i$, and R^* is the completion of R.

COROLLARY 3. *Assume that no M_i is a prime divisor of zero, and that R^* is integrally closed. Then the completion of each R_{M_i} is normal (hence R_{M_i} is a normal local domain).*

PROOF. Since no M_i is a prime divisor of zero, there is a nonzero-divisor b in the Jacobson radical of R^* [4, p. 267]. Hence by Corollary 2, $R^* = \bigoplus R^*/\mathfrak{q}_i$, where $\mathfrak{q}_i$ runs through the prime divisors of zero in R^*. Since the idempotents of the total quotient ring of R^* are in R^*, no maximal ideal in R^* contains more than one primed divisor of zero. Therefore, there are d prime divisors of zero in R^*, since $R^*/\mathfrak{q}_i$ is a complete normal local domain. Let M_iR^* be the maximal ideal in R^* which contains $\mathfrak{q}_i$. Then it is immediately seen that $R^*_{M_iR^*} = R^*/\mathfrak{q}_i \supseteq R/(\mathfrak{q}_i \cap R) = R_{M_i}$. Since R_{M_i} is a dense subspace of $R^*/\mathfrak{q}_i$ [4, p. 283], the completion of R_{M_i} is normal. It is well known [1, p. 59] that this implies that R_{M_i} is a normal local domain, q.e.d.

LEMMA 6. *Let b be a nonzero-divisor in J, let $R^{*\prime}$ be the integral closure of R^* in its total quotient ring, and let $T = R^{*\prime} \cap R^*[1/b]$. If there is an integer n such that $b^nT \subseteq bR^*$, then R is analytically unramified. Conversely, if R is analytically unramified, then for every nonzero-divisor c in R there is an integer k (depending on c) such that $c^k(R^{*\prime} \cap R^*[1/c]) \subseteq cR^*$.*

PROOF. Since b is not a divisor of zero in R, b is not a divisor of zero in R^* [4, p. 267], so $R^*[1/b]$ is contained in the total quotient ring Q of R^*. Let x be a nilpotent element in R^*. Then $x/b^i \in T$, for all $i \geqq 1$. Therefore, if $b^n T \subseteq bR^*$, then $x \in b^i T \subseteq b^{i-n+1}R^* \subseteq J^{i-n+1}R^*$, for all $i \geqq n$. Since $\bigcap J^i R^* = 0$, $x = 0$. Hence R is analytically unramified. Conversely, let R be analytically unramified and let $\mathfrak{q}_1, \cdots, \mathfrak{q}_n$ be the prime divisors of zero in R^*. Then $R^{*\prime} = \bigoplus_1^n (R^*/\mathfrak{q}_i)'$, where $(R^*/\mathfrak{q}_i)'$ is the integral closure of $R^*/\mathfrak{q}_i$. Since $(R^*/\mathfrak{q}_i)'$ is a finite $R^*/\mathfrak{q}_i$ module [**1**, p. 112], $R^{*\prime}$ is a finite R^*-module. Thus $R^{*\prime} \cap R^*[1/c]$ is a finite R^*-module, for every non-zero-divisor c in R. Hence, since every element in $R^{*\prime} \cap R^*[1/c]$ can be written in the form r/c^j, where $r \in (c^j R^*)_a$, the last statement is clear, q.e.d.

COROLLARY 4. *With the same notation as Lemma* 6, *assume* $(bR^*)_a = bR^*$. *Then R is analytically unramified.*

PROOF. If $t \in T$, then $t = r/b^j$, where $r \in (b^j R^*)_a$. Since bR^* and $b^j R^*$ have the same prime divisors, $(b^j R^*)_a = b^j R^*$ (Lemma 4). Therefore $T = R^*$, hence $bT = bR^*$, and so R is analytically unramified by Lemma 6, q.e.d.

LEMMA 7. *Let $\mathfrak{p}$ be a height one prime ideal in R. If $R_{\mathfrak{p}}$ is normal, and if $\mathfrak{p}R^* = \bigcap_1^h \mathfrak{p}_i^*$, where each $\mathfrak{p}_i^*$ is a prime ideal in R^*, then each $R^*_{\mathfrak{p}_i^*}$ is normal, and $\mathfrak{p}^{(n)}R^* = \bigcap_1^h \mathfrak{p}_i^{*(n)}$ (where $\mathfrak{q}^{(n)}$ is the nth symbolic power of a prime ideal $\mathfrak{q}$).*

PROOF. Since $R_{\mathfrak{p}}$ is a normal local domain which is not a field, $\mathfrak{p}$ is not a prime divisor of zero. Let b be an element in $\mathfrak{p}$ such that $b'R_{\mathfrak{p}} = \mathfrak{p}'R_{\mathfrak{p}}$ ($B'R_{\mathfrak{p}}$ denotes the ideal in $R_{\mathfrak{p}}$ generated by an ideal B in R). Then $0:bR \subseteq \mathfrak{q}$, where $\mathfrak{q}$ is the prime divisor of zero contained in $\mathfrak{p}$. Therefore, $(0:bR)R^* = 0R^*:bR^*$ [**4**, p. 267] $\subseteq \mathfrak{q}R^* \subset \mathfrak{p}R^* \subseteq \mathfrak{p}_i^*$ $(i = 1, \cdots, h)$. Fix i, set $\mathfrak{p}_i^* = \mathfrak{p}^*$, and let $\mathfrak{q}^*$ be a prime divisor of $0R^*$ which is contained in $\mathfrak{p}_i^*$. Then $\mathfrak{q}^* \cap R$ is a prime divisor of zero [**4**, p. 267] and is contained in $\mathfrak{p} = \mathfrak{p}^* \cap R$. Hence $\mathfrak{q}^* \cap R = \mathfrak{q}$. Further, since $\mathfrak{q}$ is the only $\mathfrak{q}$-primary ideal, every $\mathfrak{q}^*$-primary ideal contracts in R to $\mathfrak{q}$. Hence $R_{\mathfrak{p}}$ is a subring of $R^*_{\mathfrak{p}^*}$, and, since $0R^*:bR^* \subseteq \mathfrak{q}R^*$, b' is not a zero-divisor in $R^*_{\mathfrak{p}^*}$. Since $\mathfrak{p}R^*$ is semi-prime, $b'R^*_{\mathfrak{p}^*} = \mathfrak{p}'R^*_{\mathfrak{p}^*} = \mathfrak{p}^{*\prime}R^*_{\mathfrak{p}^*}$. Therefore $R^*_{\mathfrak{p}^*}$ is normal (Lemma 2 and [**3**, pp. 276–278]). The proof that $\mathfrak{p}^{(n)}R^* = \bigcap_1^h \mathfrak{p}_i^{*(n)}$ is the same as that in [**2**]. Namely, since the result is true for $n = 1$, let $n > 1$ and assume $\mathfrak{p}^{(n-1)}R^* = \bigcap_1^h \mathfrak{p}_i^{*(n-1)}$. Let c be an element in $bR:\mathfrak{p}$ which is not in $\mathfrak{p}$ (since $b'R_{\mathfrak{p}} = \mathfrak{p}'R_{\mathfrak{p}}$, $bR:\mathfrak{p} \not\subseteq \mathfrak{p}$), and let $d^* \in \bigcap_1^h \mathfrak{p}_i^{*(n)} \subset \mathfrak{p}R^*$. Since $c \in bR^*:\mathfrak{p}R^*$, $cd^* = br^*$, for some $r^* \in R^*$, hence by the choice of c and b, $b'r^{*\prime}R^*_{\mathfrak{p}_i^*} = c'd^{*\prime}R^*_{\mathfrak{p}_i^*} = d^{*\prime}R^*_{\mathfrak{p}_i^*} \subseteq \mathfrak{p}_i^{*\prime n}R^*_{\mathfrak{p}_i^*} = b'^n R^*_{\mathfrak{p}_i^*}$ $(i = 1, \cdots, h)$. Therefore, $r^* \in \bigcap_1^h \mathfrak{p}_i^{*(n-1)}$, so by induction $r^* \in \mathfrak{p}^{(n-1)}R^*$. Thus $cd^* = br^* \in \mathfrak{p}^{(n)}R^*$, hence $d^* \in \mathfrak{p}^{(n)}R^*:cR^* = (\mathfrak{p}^{(n)}:cR)R^*$ [**4**, p. 267] $= \mathfrak{p}^{(n)}R^*$, since $c \notin \mathfrak{p}$.

Thus $\bigcap_1^h \mathfrak{p}_i^{*(n)} \subseteq \mathfrak{p}^{(n)}R^*$, and since the opposite inclusion is clear, $\mathfrak{p}^{(n)}R^* = \bigcap_1^h \mathfrak{p}_i^{*(n)}$, q.e.d.

THEOREM 1. *Let R be a semi-local ring with Jacobson radical J, and let R^* be the completion of R. Assume there is a nonzero-divisor b in R such that $(bR)_a = bR$ and $\mathfrak{p}R^*$ is semi-prime, for every prime divisor $\mathfrak{p}$ of bR. Then $(bR^*)_a = bR^*$. If $b \in J$, then R is analytically unramified.*

PROOF. If b is a unit in R, then $(bR^*)_a = bR^* = R^*$. Hence assume b is a nonunit in R, and let $\mathfrak{p}_1, \cdots, \mathfrak{p}_n$ be the prime divisors of bR. Since each $R_{\mathfrak{p}_i}$ is a discrete Archimedian valuation ring (Lemmas 2 and 4), every $\mathfrak{p}_i$-primary ideal is a symbolic power of $\mathfrak{p}_i$. Therefore $bR = \bigcap_1^n \mathfrak{p}_i^{(e_i)}$, so $bR^* = \bigcap_1^n (\mathfrak{p}^{(e_i)}R^*)$ [4, p. 269]. Fix i, set $\mathfrak{p}^{(e)} = \mathfrak{p}_i^{(e_i)}$, and let $\mathfrak{p}_1^*, \cdots, \mathfrak{p}_h^*$ be the prime divisors of $\mathfrak{p}R^*$. Then $\mathfrak{p}^{(e)}R^* = \bigcap_1^h \mathfrak{p}_i^{*(e)}$ and each $R^*_{\mathfrak{p}_i^*}$ is normal (Lemma 7). Thus the prime divisors of bR^* are the prime divisors of the $\mathfrak{p}_iR^*$ $(i = 1, \cdots, h)$, hence $(bR^*)_a = bR^*$ (Lemma 4). Therefore, if $b \in J$, then by Corollary 4, R is analytically unramified, q.e.d.

COROLLARY 5. *Let R, R^* and b be as in Theorem 1, and let S^* be the integral closure of R^* in its total quotient ring. If there is an element v in S^* such that $bv \in R^*$, then $v \in R^*$.*

PROOF. $bv \in bS^* \cap R^* = (bR^*)_a = bR^*$, q.e.d.

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