FINITE SUPERSOLVABLE WREATH PRODUCTS

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1. Introduction. In [1], Baumslag found all nilpotent standard restricted wreath products $A \wr B$. It is immediate that $A \wr B$ is solvable if and only if A and B are. In this paper we prove the following results.

THEOREM. *A finite wreath product $A \wr B$ is supersolvable if and only if:*

- (i) *A is nilpotent,*
- (ii) *either B is Abelian or B' is a (nontrivial) p -group with A a p -group for the same prime p , and*
- (iii) *for each prime q dividing $o(A)$, $q \equiv 1 \pmod{m}$, where m is the exponent of B/Q , Q being the Sylow q -subgroup of B (Q is unique by (ii), and may be trivial).*

COROLLARY. *If A and B are finite and satisfy conditions (i), (ii), and (iii) above, then any extension of A by B is supersolvable.*

2. Notation and definitions. All groups considered are finite and are written multiplicatively. E denotes the trivial group and 1 is used for the identity of all groups. Other notation is standard (as found, for example, in [8]).

Let A and B be nontrivial abstract groups, and let $F = A^B$ be the direct sum of copies of A indexed by the set B . Explicitly, F is the set of all functions from B into A , made into a group by component-wise multiplication. For $f \in F$ and $b \in B$, define $f^b \in F$ by $f^b(y) = f(yb^{-1})$ for all $y \in B$. Then for each $b \in B$, the mapping $f \rightarrow f^b$ is an automorphism of F , and the group of all such automorphisms is isomorphic to B . The (standard restricted) *wreath product of A and B* , $A \wr B$, is defined to be the group generated by F and B with relations $b^{-1}fb = f^b$ for all $f \in F$, $b \in B$. Here $F \triangleleft A \wr B$, and $A \wr B$ is a splitting extension of F by the group of automorphisms B . W will be used throughout to designate $A \wr B$, F (called the *base group*) will always be as defined above, and K will denote the subgroup of W defined by $K = \{f \in F : \prod_{x \in B} f(x) \in A'\}$.

We say that a group G is *supersolvable* if and only if it has an invariant series whose factors are of prime order, that is, a series $G = G_n, G_{n-1}, \dots, G_0 = E$ with each $G_i \triangleleft G$ and each G_{i+1}/G_i of prime

order (also called a supersolvable series). G is *nilpotent* if and only if it has a series $G = G_m, G_{m-1}, \dots, G_0 = E$ in which each $G_i \triangleleft G$ and each $G_{i+1}/G_i \subseteq Z(G/G_i)$. The properties of supersolvable and nilpotent groups needed can be found in [3] or [8]. Certain of these, as well as some results on wreath products, are listed for convenience in the following lemma.

LEMMA 0.

- (0.1) $W' = KB'$ [6, Corollary 4.5].
- (0.2) If C is a subgroup of A and D a subgroup of B , then $C \wr D$ can be embedded in $A \wr B$ [7].
- (0.3) Every extension of A by B can be embedded in $A \wr B$ [5], [7].
- (0.4) $A \wr B$ is nilpotent if and only if A and B are p -groups for the same prime p [1].
- (0.5) If G is nilpotent then G is supersolvable.
- (0.6) If G is supersolvable then G' is nilpotent.
- (0.7) Subgroups of nilpotent (supersolvable) groups are nilpotent (supersolvable).
- (0.8) G is nilpotent if and only if H a proper subgroup of G implies H a proper subgroup of $N_G(H)$.
- (0.9) G is nilpotent if and only if it is the direct product of its Sylow subgroups.

3. The proofs.

LEMMA 1. If $A \wr B$ is supersolvable, then A is nilpotent.

PROOF. If A is not nilpotent, it contains a proper subgroup H with $N_A(H) = H$ (0.8). Then $H_1 = \{f \in K : f(x) \in H \text{ for all } x \in B\}$ is a proper subgroup of $K \subseteq W'$ (0.1). Suppose $f \in N_K(H_1) \setminus H_1$. Then there exists $x \in B$ such that $f(x) = a \in A \setminus H$. If we choose $y \in B$, $y \neq x$, and define $g \in H_1$ by $g(x) = h$, $g(y) = h^{-1}$, and $g(z) = 1$ for $z \neq x, y$, where h is any element of H , we see that $f^{-1}gf \in H_1$, so that $(f^{-1}gf)(x) = a^{-1}ha \in H$. But since h is arbitrary in H this implies $a \in N_A(H) \setminus H$, which is impossible. Therefore $N_K(H_1) = H_1$, contradicting the nilpotency of K (0.8) and W' (0.7) and therefore the supersolvability of W (0.6).

LEMMA 2. If $A \wr B$ is supersolvable, then B is Abelian or B' is a (nontrivial) p -group and A is a p -group for the same prime p .

PROOF. If the lemma is false, then there are primes $p \mid o(B')$, $q \mid o(A)$, $p \neq q$. Take $a \in A$ with $o(a) = q$ and $b \in B'$ with $o(b) = p$. Choose arbitrary distinct elements $x, y \in B$, and define $f \in F$ by $f(x) = a$, $f(y) = a^{-1}$, $f(z) = 1$ for $z \neq x, y$. Then $f \in K \subseteq W'$, $b \in W'$, $o(f) = q$, $o(b) = p$, but $fb \neq bf$. (If $a = a^{-1}$ for all $a \in A$, then $o(B) > 2$ and

x and y can be chosen so that we have $fb \neq bf$.) Therefore W' is not nilpotent (0.9) and so W is not supersolvable (0.6).

LEMMA 3. *A wr B is supersolvable if and only if A is nilpotent and J_p wr B is supersolvable for each p dividing $o(A)$, where J_p denotes the cyclic group of order p .*

PROOF. If A is not nilpotent, W is not supersolvable by Lemma 1, while if $p \mid o(A)$ and J_p wr B is not supersolvable, then A wr B is not supersolvable by (0.2) and (0.7).

For sufficiency, induct on $o(A)$. There is a subgroup H of $Z(A)$ of prime order. Since H wr B is supersolvable, there is an invariant series of W up through H^B with factors of prime order. Then $W/H \cong (A/H)$ wr B is supersolvable by induction, and therefore W is also supersolvable.

LEMMA 4. *If $p \nmid n$, p prime, then J_p wr J_n is supersolvable if and only if $p \equiv 1 \pmod{n}$.*

PROOF. Since J_n is supersolvable and $W/F \cong J_n$, and F is Abelian, it is necessary and sufficient to show that F has a series $F = F_n, F_{n-1}, \dots, F_0 = E$ with each $o(F_{i+1}/F_i) = p$ and each F_i invariant under the action of $B = J_n$. Here we may view F as a vector space of dimension n over $GF(p)$, a field with p elements, acted on by the cyclic linear transformation $\bar{b}$ induced by b , where $\langle b \rangle = B = J_n$. The minimum polynomial of b is $\lambda^n - 1$, and the subspaces invariant under $\bar{b}$ may be put in one-one correspondence with the factors of $\lambda^n - 1$ which have leading coefficient 1, with one such subspace containing another if and only if the factor corresponding to the first divides the factor corresponding to the second [4, p. 129]. Therefore J_p wr J_n is supersolvable if and only if $\lambda^n - 1$ splits into linear factors over $GF(p)$. Since $p \nmid n$, $\lambda^n - 1$ has no repeated roots in $GF(p)$, and so $\lambda^n - 1$ splits into linear factors if and only if $n \mid (p-1)$, for the roots of $\lambda^n - 1$ form a subgroup of the multiplicative group of $GF(p)$.

LEMMA 5. *If B is Abelian and $p \nmid o(B)$, then J_p wr B is supersolvable if and only if $p \equiv 1 \pmod{m}$, where $m = \text{Exp } B$.*

PROOF. If $p \not\equiv 1 \pmod{m}$, we may apply (0.2), (0.7), and Lemma 4 to conclude that J_p wr B is not supersolvable since J_p wr J_m is not. As in the proof of Lemma 4, it is now sufficient to show that if $p \equiv 1 \pmod{m}$ then F has a series $F = F_n, F_{n-1}, \dots, F_0 = E$ with each $o(F_{i+1}/F_i) = p$ and each F_i invariant under the action of B , where in the present case $n = o(B)$. We interpret B as a group acting on F , a vector space of dimension n over $GF(p)$, and apply Maschke's

Theorem [2, pp. 40–41] to conclude that F is a direct sum of irreducible B -subspaces. If all of these irreducible B -subspaces are of dimension one, it is clear how to construct the sequence $F = F_n, F_{n-1}, \dots, F_0 = E$. However, since $p \nmid \text{Exp } B$, it follows [2, Theorem 9.10, p. 37] that B has $o(B)$ distinct one-dimensional $GF(p)$ -representations, and therefore all irreducible B -subspaces are one-dimensional [2, p. 213]. (Since $p \equiv 1 \pmod{m}$, $\lambda^m - 1$ splits into linear factors over $GF(p)$, and so in the notation of [2], $\sqrt[m]{1} \in GF(p)$.)

LEMMA 6. *If P is a Sylow p -subgroup of B containing B' , then $J_p \text{ wr } B$ is supersolvable if and only if $p \equiv 1 \pmod{m}$, where $m = \text{Exp } (B/P)$.*

PROOF. Suppose $J_p \text{ wr } B$ is supersolvable. Since $P \triangleleft B$, P has a complement Q in B by Schur's splitting theorem [8, 9.3.6]. Then $J_p \text{ wr } Q$ is supersolvable by (0.2) and (0.7), and hence $p \equiv 1 \pmod{m}$ by Lemma 5. Now assume $p \equiv 1 \pmod{m}$. We can use the fact that $P \triangleleft B$ to conclude that $FP \triangleleft W$, and thereby think of W as an extension of FP by B/P . Then Lemmas 3 and 5 and (0.3) and (0.7) apply to yield that $J_p \text{ wr } B$ is supersolvable.

PROOF OF THEOREM. If $A \text{ wr } B$ is supersolvable, the necessity of (i) follows from Lemma 1, that of (ii) from Lemma 2, and that of (iii) from Lemma 3 and Lemma 6. Sufficiency is shown by Lemmas 3, 5, 6, and (0.4).

PROOF OF COROLLARY. Apply the Theorem, (0.3), and (0.7).

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