Bibliography

1. J. Dixmier, *Les algèbres d'opérateurs dans l'espace Hilbertien*, Gauthier-Villars, Paris, 1957.
2. I. C. Gohberg and M. G. Kreĭn, *Fundamental theorems on defect numbers, etc.*, Uspehi Mat. Nauk **12** (1957), 43–118.
3. P. R. Halmos, *An introduction to Hilbert space*, Chelsea, New York, 1951.
4. E. Hille and R. S. Phillips, *Functional analysis and semi-groups*, Amer. Math. Soc. Colloq. Publ. Vol. 31, Amer. Math. Soc., Providence, R. I., 1957.
5. G. Lumer and M. Rosenblum, *Linear operator equations*, Proc. Amer. Math. Soc. **10** (1959), 32–41.
6. R. Schatten, *Norm ideals of completely continuous operators*, Ergebnisse der Mathematik, Berlin, 1960.

University of Michigan

KIRZBRAUN'S THEOREM AND KOLMOGOROV'S PRINCIPLE[1]

EDWARD SILVERMAN

Let B be a Banach space. A *distance function* p on B is a nonnegative valued function which is continuous, positively homogeneous of degree one and subadditive. If A is a set and if x and y map A into B then we write xpy if $p(x(a)-x(b))\leqq p(y(a)-y(b))$ for all $a, b\in A$. If A is a k-cell, if B is Euclidean space, if p is the norm and if L is Lebesgue area, then Kolmogorov's Principle, K.P., asserts that $Lx\leqq Ly$ if xpy [**H.M.**]. Lebesgue area is a parametric integral of the type considered by McShane [**M**], for smooth enough maps. In this paper we consider other such integrals, not necessarily symmetric, for which a type of K.P. holds. We conclude with a minor application to a Plateau problem.

The proof of K.P. follows from

Kirzbraun's Theorem. *If $A\subset E^n$ and $t\colon A\to E^n$ is Lipschitzian, then there exists an extension T of t, $T\colon E^n\to E^n$, and T is Lipschitzian with the same constant as t* [**S**].

The proof of the version of K.P. in which we are interested depends upon an embedding of E^n in m, the space of bounded sequences [**B**],

---

Received by the editors January 11, 1965.

[1] This research was supported in part by National Science Foundation Grant No. GP634.

and an extension theorem, resembling that of Kirzbraun, for m.

Let α be the distance function on m defined by $\alpha(a) = \max\{\sup a^i, 0\}$. In a manner to be made precise in the Embedding Theorem, α is universal as a distance function.

Let B be a separable Banach space and p be a distance function on B. Let $\{b_i\}$ be dense on ∂K where $K = \{b \in B \mid p(b) \leqq 1\}$. There exist, by the Hahn-Banach Theorem [**B**], $f_i \in B^*$ such that $f_i(b_i) = 1$ and $f_i(b) \leqq p(b)$ for all $b \in B$. Since p is continuous there exists $N' > 0$ such that $\|f_i\| \leqq N'$ for all i.

EMBEDDING THEOREM [**S3**]. *Let* $Vb = \{f_i(b)\}$. *Then* $V \in L(B, m)$ *and* $p = \alpha V$.

The proof is almost immediate.

Let N be the set of natural numbers. If $k \in N$, then $\Lambda^k m$ is the space of all bounded real-valued anti-symmetric functions on N^k with the sup norm.

If $a = (a_1, \cdots, a_k) \in m^k$ let $U_a \in L(E^k, m)$ and $\Lambda a = a_1 \Lambda \cdots \Lambda a_k \in \Lambda^k m$ be defined by $U_a h = \sum_{i=1}^k h_i a_i$ for all $h = (h_1, \cdots, h_k) \in E^k$, and

$$(\Lambda a)(n_1, \cdots, n_k) = \det \begin{vmatrix} a_1^{n_1} & \cdots & a_1^{n_k} \\ \cdot & \cdots & \cdot \\ a_k^{n_1} & \cdots & a_k^{n_k} \end{vmatrix}$$

where, of course, $a_i = (a_i^1, a_i^2, \cdots)$. Furthermore, if $\zeta = (\zeta_1, \cdots, \zeta_k) \in m^{*k}$ then

$$[\Lambda a, \zeta_1 \Lambda \cdots \Lambda \zeta_k] = \det \begin{vmatrix} \zeta_1(a_1) & \cdots & \zeta_k(a_1) \\ \cdot & \cdots & \cdot \\ \zeta_1(a_k) & \cdots & \zeta_k(a_k) \end{vmatrix}.$$

If $E^k \subset m$ and $a_1, \cdots, a_k \in E^k$, then $\|\Lambda a\|$ is the volume of the parallelepiped spanned by $a_1, \cdots, a_k$. If similarly, $b \in (E^k)^k \subset m^k$ then $\|\Lambda a\| \leqq \|\Lambda b\|$ if $U_a \alpha U_b$, and this fact is vital for the validity of K.P. In general, we write $a \prec b$ if $U_a \alpha U_b$.

Let M be the set of all distance functions on $\Lambda^k m$ and let $M' = \{f \in M \mid f(\Lambda a) \leqq f(\Lambda b) \text{ whenever } a \prec b\}$.

Let Q be a k-cell. If $x \in C(Q, m)$ is Lipschitzian, then $d_i x = \{\partial x^j / \partial u^i\}$ exists almost everywhere. We write dx for $(d_1 x, \cdots, d_k x) \in m^k$ and, as above, Λdx for $d_1 x \Lambda \cdots \Lambda d_k x$.

Suppose that B is a Banach space contained in m. Then we can identify $\Lambda^k B$ with the appropriate subspace of $\Lambda^k m$. Let Q be a k-cell and $P(Q, B)$ be the subset of $C(Q, B)$ consisting of quasilinear functions. Then $P(Q, B)$ is dense in $C(Q, B)$. If $z\in P(Q, B)$ and $f\in M$ then we define $\mathcal{E}_f z = \sum f(\Lambda dz)\cdot \mathrm{vol}\,\Delta$ where the summation is taken over the oriented simplexes, Δ, of linearity of z. Let $M'' = \{f\in M' \mid \mathcal{E}_f$ is lower semi-continuous on $P(Q, B)\}$. If $f\in M''$ then the Fréchet extension, $L_{f,B}$, of $\mathcal{E}_f$ is the Lebesgue area: $L_{f,B}x = \liminf_{z\to x} \mathcal{E}_f z$ for each $x\in C(Q, B)$ where $z\in P(Q, B)$. Let us write L_f for $L_{f,m}$. We will make use of the fact, proved like the two-dimensional case in [**S3**], that $L_{f,B}x = \int f(\Lambda dx) = L_f x$ for x in a subset of $C(Q, B)$ which contains, in particular, all Lipschitzian maps.

It is obvious that $L_f \leqq L_{f,B}$, but not at all evident that the equality holds, though this known for $k=2$ and for a wide class of functions for $k>2$.

Let e be the identity mapping on m.

Extension Theorem. *Let $\sigma>0$, $A\subset m$ and $t\colon A\to m$ with $t\alpha(\sigma e|A)$, i.e., $\alpha(ta-tb)\leqq \sigma\alpha(a-b)$ for all $a, b\in A$. Then there exists an extension T of t such that $T\alpha(\sigma e)$. Let $\psi_i(a, c) = (ta)^i + \sigma\alpha(c-a)$ for all $a\in A$ and $c\in m$, and let $T_i(c) = \inf\{\psi_i(a, c) \mid a\in A\}$. Then we can let $Tc = \{T_i(c)\}$.*

The proof is like that of the corresponding theorem in [**S3**]. First suppose that $c\in A$. Then $\psi_i(a, c) - \psi_i(c, c) = (ta)^i + \sigma\alpha(c-a) - (tc)^i \geqq \sigma\alpha(c-a) - \alpha(tc-ta) \geqq 0$ and so $T|A = t$. If $b\in m$ then $\psi_i(a, b) - \psi_i(a, c) = \sigma[\alpha(b-a) - \alpha(c-a)] \leqq \sigma\alpha(b-c)$ and so $|\psi_i(a,b) - \psi_i(a,c)| \leqq \sigma\|b-c\|$. Thus $|T_i(b)| \leqq \|tc\| + \sigma\|b-c\|$ and $Tb\in m$ for all b. If $d\in m$ then $[T_i(b) - T_i(d)] \leqq \sup[\sigma\alpha(b-a) - \sigma\alpha(d-a)] \leqq \sigma\alpha(b-d)$, and the proof is complete.

Let $\pi_n a = b$ where $b^i = a^i$ if $i\leqq n$ and $b^i = 0$ for $i>n$.

It is clear that $x_n\to y$, in $C(Q, m)$, if and only if

$$\max\{\alpha(x_n(p) - y(p)) \mid p\in Q\}\to 0 \text{ and } \max\{\alpha(y(p) - x_n(p)) \mid p\in Q\}\to 0.$$

Hence $Tx_n\to Ty$ whenever $x_n\to y$ and $T\alpha e$.

Kolmogorov's Principle. *If $x, y\in C(Q, m)$, if $x\;\alpha\;y$ and if $f\in M''$, then $L_f x\leqq L_f y$.*

Let $z\in P(Q, m)$ and $T\alpha e$. Let $w = Tz$ and $w_n = \pi_n w$. Then w is Lipschitzian. Hence for almost all $p\in Q$, all $h\in R^k$ and $s>0$, $\alpha(sdw_n(p)\cdot h + o(s)) = \alpha(w_n(p+sh) - w_n(p)) \leqq s\alpha(dz(p)\cdot h)$. It follows that $dw(p) \prec dz(p)$ and, since $f\in M''$, $L_f(Tz) = \int f(\Lambda dw) \leqq \int f(\Lambda dz) = \mathcal{E}_f z$.

If we let $ty = x$ then $t\alpha(e|\text{range } y)$ and, by the Extension Theorem, there exists $T\alpha e$ with $Ty = x$. Let $z_n\to y$ with $z_n\in P(Q, m)$ and $\mathcal{E}_f z_n \to L_f y$. Then $Tz_n\to x$ and $L_f x \leqq \liminf L_f(Tz_n) \leqq \lim \mathcal{E}_f z_n = L_f y$.

If seems appropriate to show that there exist $A\in M''$ such that A is not symmetric.

If $a \in m^k$, then R_a, the range of U_a, is the vector subspace of m spanned by the components of a and R_a^* is the space of linear functionals over R_a. If $\zeta \in R_a^*$, let $N_a(\zeta) = \inf\{K \mid K\alpha(c) \geqq \zeta(c)$ for all c in $R_a\}$. Let $A(\Lambda a) = \sup\{[\Lambda a, \zeta_1 \Lambda \zeta_2 \Lambda \cdots \Lambda \zeta_k] \mid N_a(\zeta_i) \leqq 1\}$. Suppose that $a \prec b$ and $T \in L(R_b, R_a)$ is defined by $U_a = T \circ U_b$. Let $T^* \in L(R_a^*, R_b^*)$ be defined by $T^*\zeta = \zeta \circ T$. If $K\alpha \circ U_a \geqq \zeta \circ U_a$, then $K\alpha \circ U_b \geqq T^*\zeta \circ U_b$ and $N_b(T^*\zeta) \leqq N_a(\zeta)$. Thus

$$\begin{aligned} A(\Lambda a) &= \sup\{[\Lambda b, T^*\zeta_1 \Lambda \cdots \Lambda T^*\zeta_k] \mid N_a(\zeta_i) \leqq 1\} \\ &\leqq \sup\{[\Lambda b, \eta_1 \Lambda \cdots \Lambda \eta_k] \mid N_b(\eta_i) \leqq 1\} = A(\Lambda b). \end{aligned}$$

For each $\gamma \in \Lambda^k m$ we set

$$A(\gamma) = \inf\left\{ \sum_{i=1}^{p} A(\Lambda a_i) \,\middle|\, \sum_{i=1}^{p} (\Lambda a_i)(n) \geqq \gamma(n) \text{ for all } n \in N^k \right\}.$$

Then $A \in M'$. It can be shown, by making suitable modifications in the argument of [S3], that $A \in M''$.

We conclude with an interesting, though trivial, application. If $f \in M''$ and $\phi \in C(\partial Q, m)$ and $B(\phi) = \inf\{L_f x \mid x \in C(Q, m)$ and $x \mid \partial Q = \phi\}$. Then $B(\phi) \leqq B(\phi')$ if $\phi \alpha \phi'$. The proof goes as follows: There exists $T\alpha e$ with $\phi = T\phi'$. Let $y \in C(Q, m)$ with $y \mid \partial Q = \phi'$. Then $Ty \in C(Q, m)$ and $Ty \mid \partial Q = \phi$. Hence $B(\phi) \leqq L_f(Ty) \leqq L_f y$.

References

B. S. Banach, *Theorie des operations lineaires*, Monografje Matematyczne, Warsaw, 1932.

H.M. R. G. Helsel and E. J. Mickle, *The Kolmogoroff principle for the Lebesgue area*, Bull. Amer. Math. Soc. **54** (1948), 235–238.

M. E. J. McShane, *Existence theorems for double integral problems of the calculus of variations*, Trans. Amer. Math. Soc. **38** (1935), 549–563.

S. I. J. Schoenberg, *On a theorem of Kirzbraun and Valentine*, Amer. Math. Monthly **60** (1953), 620–622.

S1. E. Silverman, *A problem of least area*, Pacific J. Math. **14** (1964), 309–331.

S2. ———, *Geodesics and Lebesgue area*, Proc. Amer. Math. Soc. **15** (1964), 775–780.

S3. ———, *Definitions of Lebesgue area for surfaces in metric spaces*, Riv. Mat. Univ. Parma **2** (1951), 47–76.

PURDUE UNIVERSITY