ELEMENTARY PROOF OF HU'S THEOREM ON ISOTONE MAPPINGS

N. C. HSU

1. **Introduction.** Let E be a partially ordered set of finite order, and let m, n be two natural numbers. We pose the following questions: Find a necessary and sufficient condition under which there exists a mapping f of E into a linearly ordered set $L=\{1<2< \cdots <t\}$ such that

(a) f is strictly isotone in the sense that $a<b$ in E implies that $f(a)<f(b)$ in L.

(b) The cardinal number of $f^{-1}(n)$ is not greater than m for every $n \in L$.

For the case where E satisfies the condition that every element of E is covered by at most a single element, a simple and elegant answer was given by Hu [2]. His proof is, however, far from simple. The purpose of this note is to provide a much simpler and more transparent proof of his theorem.

2. **Preliminaries.** Let E be a partially ordered set of finite order. We define the height of an element in E and the height of E in a usual way (see, e.g. [1]). By the depth of an element x in E, we mean the height of the element $\hat{x}$ in the dual $\hat{E}$ of E. By $h(E)$, we denote 1 plus the height of E. By E_i and E^j, we denote the set of all elements of depth $i-1$ in E and the set of all elements of height $j-1$ in E, respectively. For example, $E_{h(E)}$ is the set of all elements of maximum depth, and E^1 is the set of all minimal elements. Evidently, $E_{h(E)} \subseteq E^1$ is valid. We denote $E_i \cap E^j$ by E_i^j, and the cardinal number of E by $|E|$. Finally, we put

$$w_i(E) = |E_{h(E)}| + |E_{h(E)-1}| + |E_{h(E)-2}| + \cdots + |E_{h(E)+1-i}|,$$

for $i=1, 2, 3, \cdots, h(E)$. What makes our proof so simple is the following

DEFINITION. Let m and t be two positive integers. E is called (m, t) *bounded* if and only if the following inequalities are satisfied:

$$w_i(E) \leqq (i+t-h(E))m, \qquad \text{for } i = 1, 2, 3, \cdots, h(E).$$

Our goal is to give a proof of

THEOREM [2, p. 847]. *Let E be a partially ordered set of finite order*

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in which every element is covered by at most a single element. Then, E is (m, t) bounded if and only if there exists a strictly isotone mapping f of E into the linearly ordered set $L=\{1<2<\cdots<t\}$ such that the cardinal number of $f^{-1}(n)$ is not greater than m for every $n\in L$.

3. **Proof of the theorem.** We begin with a constructive proof of

LEMMA 1. *If a partially ordered set E is (m, t) bounded and also satisfies the condition*

(0) $|E_i|>m$ *for some i implies that for the same i*

$$|E_i^1|+|E_{i+1}^1|+|E_{i+2}^1|+\cdots+|E_{h(E)}^1|\geqq m,$$

then there exists a subset $F\subseteq E$ such that

(1) $|E-F|\leqq m$,

(2) *either $E-F\subseteq E_1$ or $E-F\subseteq E^1$, and*

(3) *F is $(m, t-1)$ bounded.*

PROOF. In case $|E_1|\leqq m$, let $F=E-E_1$. Then $h(F)=h(E)-1$ and

$$\begin{aligned} w_i(F) &= w_i(E) \\ &\leqq (i+t-h(E))m \\ &= (i+t-1-h(F))m, \end{aligned}$$

for $i=1, 2, 3, \cdots, h(F)$, and all three requirements are satisfied. In case $|E_1|>m$, we have $|E^1|\geqq m$ by the condition (0), and two subcases are conceivable; either $|E_{h(E)}^1|>m$ or $|E_{h(E)}^1|\leqq m$. If $|E_{h(E)}^1|>m$, let $F=E-G$ where G is a set of m elements taken arbitrarily from $E_{h(E)}^1$. Then we have $h(F)=h(E)$ and

$$\begin{aligned} w_i(F) &= w_i(E)-m \\ &\leqq (i+t-h(E)-1)m \\ &= (i+t-1-h(F))m, \end{aligned}$$

for $i=1, 2, 3, \cdots, h(F)$, and all three requirements are satisfied. On the other hand, if $|E_{h(E)}^1|\leqq m$, let i_0 be the integer satisfying both

$$|E_{i_0}^1|+|E_{i_0+1}^1|+|E_{i_0+2}^1|+\cdots+|E_{h(E)}^1|\geqq m$$

and

$$|E_{i_0+1}^1|+|E_{i_0+2}^1|+\cdots+|E_{h(E)}^1|<m.$$

Needless to say, we have $i_0=h(E)$ when $|E_{h(E)}^1|=m$. Now, let

$$F = E - (E^1_{i_0+1} \cup E^1_{i_0+2} \cup \cdots \cup E^1_{h(E)} \cup G),$$

where G is a set of $m-(|E^1_{i_0+1}|+|E^1_{i_0+2}|+\cdots+|E^1_{h(E)}|)$ elements taken arbitrarily from $E^1_{i_0}$. Then $h(F)=h(E)-1$ and for $i=h(E)-i_0+1,\ h(E)-i_0+2,\ h(E)-i_0+3, \cdots, h(F)$, we have

$$\begin{aligned} w_i(F) &= w_{i+1}(E) - m \\ &\leqq (i+1+t-h(E)-1)m \\ &= (i+t-1-h(F))m. \end{aligned}$$

It remains to show that

$$w_i(F) \leqq (i+t-1-h(F))m$$

for $i=1, 2, 3, \cdots, h(E)-i_0$. Suppose on the contrary that there exists a j_0, $1 \leqq j_0 \leqq h(E)-i_0$, such that

$$\begin{aligned} w_{j_0}(F) &> (j_0+t-1-h(F))m \\ &= (j_0+t-h(E))m. \end{aligned}$$

Assume that j_0 is the smallest integer having this property. Then

$$|F_{h(F)+1-j_0}| = |F_{h(E)-j_0}| > m$$

from which we have $|E_{h(E)-j_0}|>m$ and therefore by the condition (0),

$$|E^1_{h(E)-j_0}| + |E^1_{h(E)-j_0+1}| + \cdots + |E^1_{h(E)}| \geqq m.$$

Hence $h(E)-j_0 \leqq i_0$ which implies $i_0=h(E)-j_0$. Consequently,

$$\begin{aligned} w_{j_0+1}(E) &= w_{j_0}(F) + m \\ &> (j_0+1+t-h(E))m, \end{aligned}$$

contrary to the assumption that E is (m, t) bounded.

LEMMA 2 [2, p. 844]. *Let E be a partially ordered set of finite order which is not (m, t) bounded. Then, there does not exist a strictly isotone mapping f of E into the linearly ordered set $L=\{1<2<\cdots<t\}$ such that the cardinal number of $f^{-1}(n)$ is not greater than m for every $n \in L$.*

PROOF. If E is not (m, t) bounded, there exists an i such that

$$w_{h(E)-i+1} > (t+1-i)m.$$

Suppose on the contrary that there exists a mapping f described in the statement. Then, since each of $f^{-1}(1), f^{-1}(2), \cdots, f^{-1}(t+1-i)$ consists of at most m elements, there must exist an $x_i \in E_i$ such that

$f(x_i) > t+1-i$. Since f is strictly isotone, there must exist an $x_1 \in E_1$ such that $f(x_1) > t$, contrary to the assumption that $f(x) \leqq t$ for every $x \in E$.

PROOF OF THE THEOREM. Suppose that E is (m, t) bounded. Since every element of E is covered by at most a single element, the condition (0) in Lemma 1 is satisfied by every subset of E, and a desired mapping can be constructed, by finite induction, by means of Lemma 1. The converse is an immediate consequence of Lemma 2.

UNSOLVED PROBLEM. Let m and t be two positive integers. Find a necessary and sufficient condition under which there exists a strictly isotone mapping f of a more general partially ordered set E into the linearly ordered set $L = \{1 < 2 < \cdots < t\}$ such that the cardinal number of $f^{-1}(n)$ is not greater than m for every $n \in L$.

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STATE UNIVERSITY OF NEW YORK AT BUFFALO