NILPOTENT ELEMENTS IN RINGS OF INTEGRAL REPRESENTATIONS

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1. **Introduction.** Let G be a finite group, and let R be a discrete valuation ring of characteristic zero, with maximal ideal $P=\pi R$, and whose residue class field $\bar{R}=R/P$ has characteristic $p\neq 0$. By an RG-module we mean always a left RG-module which is finitely generated over R, though not necessarily R-torsionfree. Assume that the Krull-Schmidt theorem is valid for RG-modules; this is certainly the case when R is complete, or when R is a valuation ring in an algebraic number field which is a splitting field for G.

In a recent paper [**4**] we introduced the *integral representation ring*, denoted by $A(RG)$, defined as the additive group generated by the symbols $\{M\}$, one for each isomorphism class of R-torsionfree RG-modules, with relations $\{M\oplus N\}=\{M\}+\{N\}$. Multiplication in $A(RG)$ is defined by taking tensor products of modules.

The question arises as to whether the commutative ring $A(RG)$ contains any nonzero nilpotent elements. This is of special interest in view of recent results of Green [**2**] and O'Reilly [**3**], who showed that if k is a field of characteristic p, and if G has a cyclic p-Sylow subgroup, then $A(kG)$ has no nonzero nilpotent elements.

In contradistinction to this, we proved in [**4**]:

THEOREM 1. *Let G^* be a cyclic group of order n, and suppose that the Krull-Schmidt theorem holds for RG^*-modules. Assume that $n\in P^2$, and if $2\in P$ assume further that $n\in 2P$. Then $A(RG^*)$ contains at least one nonzero nilpotent element.*

The aim of the present note is to establish the following generalization.

THEOREM 2. *Suppose that the group G contains a cyclic subgroup G^* satisfying the hypotheses of Theorem 1, and assume that the Krull-Schmidt theorem holds for RG-modules. Then $A(RG)$ contains at least one nonzero nilpotent element.*

We shall use the following notation. For M an RG-module, set $\bar{M}=M/PM$. Denote by M_H the RH-module obtained from M by

Presented to the Society, November 23, 1964; received by the editors November 21, 1964.

restriction of operators, where H is a subgroup of G. For N an RH-module, let N^G denote the induced RG-module defined by

$$N^G = RG \otimes_{RH} N.$$

The *trivial* RG-module is R itself, on which each $g \in G$ acts as identity operator. If M, N are RG-modules, the notation $M|N$ means that M is isomorphic to an RG-direct summand of N.

As general reference for the techniques and definitions used in this note, we refer the reader to [**1**].

2. **Preliminaries to the proof.** Suppose hereafter that the hypotheses of Theorem 2 are satisfied, so that G contains a cyclic subgroup G^* of order n. If p is the unique rational prime contained in P, then the assumptions about n readily imply that $p^e | n$, where

$$e = \begin{cases} 2, & p = 2, \\ 1, & p \text{ odd}, p \in P^2, \\ 2, & p \text{ odd}, p \notin P^2. \end{cases}$$

Hence G^* contains a cyclic subgroup H of order p^e. Since the Krull-Schmidt theorem is assumed valid for RG^*-modules, it also holds for RH-modules. Note that $p^e \in P^2$, and if $p=2$, then $p^e \in 2P$.

Let I denote the augmentation ideal of RH, so that

$$I = \sum_{h \in H} R(h - 1).$$

Then

$$\bar{I} = \sum_{h \in H} \bar{R}(h - 1) \cong \bar{R}[x]/(x - 1)^{p^e - 1},$$

where the generator of the cyclic group H acts on the right-hand module as multiplication by x. This shows that $\bar{I}$ is indecomposable, whence so is I.

We shall show next that if K is a proper subgroup of H, and $\bar{M}$ is any $\bar{R}K$-module, then $\bar{I} | \bar{M}^H$ is impossible. For if such a relation were true, we could assume without loss of generality that $\bar{M}$ is an indecomposable $\bar{R}K$-module. Since K is cyclic, the indecomposable $\bar{R}K$-modules may be listed explicitly, and have K-dimensions $1, 2, \cdots, [K:1]$. If $\bar{M}$ has dimension $[K:1]$, then $\bar{M} = \bar{R}K$, and in that case $\bar{I} | \bar{R}H$. This is impossible since $\bar{R}H$ is indecomposable. On the other hand, if $\dim \bar{M} < [K:1]$, then $\dim \bar{M}^H = [H:K] \cdot \dim \bar{M} < p^e - 1 = \dim \bar{I}$, so also in this case $\bar{I}$ cannot be a direct summand of $\bar{M}^H$.

As in [**4**], define the RH-modules X and Y by

$$X = \pi \cdot RH + I, \qquad Y = \pi \cdot RH + R \cdot \sum_{h \in H} h.$$

Then X is a nonsplit extension of the factor module R (with trivial action of H) by the submodule I, and hence X is indecomposable. On the other hand, we showed in [4] that Y is an extension of R by a submodule J, where $\bar{R} | \bar{J}$. From these facts we were able to conclude that

$$\bar{X} \cong \bar{Y} \cong \bar{R} \oplus \bar{I}, \qquad X \text{ not isomorphic to } Y.$$

Furthermore, there exist exact sequences

$$\begin{aligned} &0 \to X \to RH \to \bar{R} \to 0 \\ &0 \to Y \to RH \to \bar{I} \to 0. \end{aligned} \tag{1}$$

Let r be a positive integer such that $p \nmid r$, and let M be any RH-module. Define a new RH-module M_r consisting of the same elements as M, but with a different action of H, namely, an element $h \in H$ acts on M_r in the same way that h^r acts on the original module M. Clearly, if M is an extension of A by B, then M_r is an extension of A_r by B_r. Further, if $R | M$ then also $R | M_r$.

We claim that $X \cong Y_r$ is impossible. To prove this, note first of all that Y_r is an extension of R by J_r. If $X \cong Y_r$ then $I \cong J_r$, and so $\bar{I} \cong \bar{J}_r$. This cannot hold true because $\bar{R} | \bar{J}$, so that $\bar{R} | \bar{J}_r$, while on the other hand $\bar{R} \nmid \bar{I}$. We have thus established our claim.

3. **Proof of the main result.** Keeping the notation of the preceding section, we are now ready to prove Theorem 2. Let us define $U = X^G$, $V = Y^G$. Since $(RH)^G \cong RG$, from the exact sequences in (1) we obtain a new pair of exact sequences

$$\begin{aligned} &0 \to U \to RG \to \bar{R}^G \to 0, \\ &0 \to V \to RG \to \bar{I}^G \to 0. \end{aligned}$$

Since $\bar{X} \cong \bar{Y}$ we conclude that $\bar{U} \cong \bar{V}$.

For any RG-module M of R-rank m, there is an exact sequence

$$0 \to U \otimes_R M \to RG \oplus \cdots \oplus RG \to \bar{R}^G \otimes_{\bar{R}} \bar{M} \to 0,$$

where m summands occur in the center module. This implies by Schanuel's Lemma that the module $U \otimes_R M$ depends only upon $\bar{M}$. Hence we may conclude that

$$U \otimes_R U \cong U \otimes_R V \cong V \otimes_R U \cong V \otimes_R V,$$

and therefore that $\{U\} - \{V\}$ has square zero in $A(RG)$.

In order to complete the proof of Theorem 2, it suffices to show that $\{U\}-\{V\}$ is nonzero, that is, that U and V are not isomorphic. Suppose that $U \cong V$, so that $X^G \cong Y^G$. Then

$$(X^G)_H \cong (Y^G)_H \quad \text{as } RH\text{-modules.}$$

However, $X \mid (X^G)_H$, and so also $X \mid (Y^G)_H$. Let us apply Mackey's Subgroup Theorem to the module $(Y^G)_H$. This yields

$$(Y^G)_H \cong \sum_g{}^{\oplus} (g \otimes Y)_K{}^H.$$

In this formula, g ranges over a full set of representatives of the (H, H)-double cosets of G. For each such g, K is the subgroup of H given by $K = H \cap gHg^{-1}$. The RK-module $g \otimes Y$ is a subspace of $RG \otimes Y$, and the action of K on $g \otimes Y$ is given by

$$ghg^{-1}(g \otimes y) = g \otimes hy, \qquad h \in H, \quad y \in Y.$$

Since $X \mid (Y^G)_H$ and X is indecomposable, we conclude that $X \mid (g \otimes Y)_K{}^H$ for some g, and hence that $\overline{X} \mid (g \otimes \overline{Y})_K{}^H$. By the remarks in §2, this cannot occur if K is a proper subgroup of H. On the other hand, suppose that $K = H$, so that $gHg^{-1} = H$. If h is a generator of the cyclic group H, we may write $g^{-1}hg = h^r$, where $p \nmid r$. Then the $\overline{R}H$-modules $g \otimes \overline{Y}$ and $\overline{Y}_r$ are isomorphic, and if $\overline{X} \mid (g \otimes \overline{Y})$, then $\overline{X} \cong \overline{Y}_r$. This is impossible by the results of §2. We have thus shown that U and V are nonisomorphic, which completes the proof of the theorem.

COROLLARY. *Let R_0 be a valuation ring in an algebraic number field, with maximal ideal P_0. Suppose that G contains a cyclic sub-group G^* of order n, where $n \in P_0^2$, and if $2 \in P_0$, assume further that $n \in 2P_0$. Then $A(R_0G)$ contains at least one nonzero nilpotent element.*

PROOF. Even though the Krull-Schmidt theorem may not hold for R_0G-modules, we may nevertheless form the representation ring $A(R_0G)$ defined as above. If L and L' are R_0G-modules, it is easily seen that $\{L\} = \{L'\}$ in $A(R_0G)$ if and only if there exists an R_0G-module M such that $L \oplus M \cong L' \oplus M$.

The construction given in Theorem 2, with R replaced by R_0, yields a pair of R_0G-modules U_0, V_0 such that $U = R \otimes U_0$, $V = R \otimes V_0$. However, the map $A(R_0G) \to A(RG)$ defined by $L \to R \otimes_{R_0} L$ is a monomorphism. Therefore $\{U_0\} - \{V_0\}$ is a nonzero nilpotent element of $A(R_0G)$, and the Corollary is proved.

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NOTE ON ANALYTICALLY UNRAMIFIED SEMI-LOCAL RINGS

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All rings in this paper are assumed to be commutative rings with a unit element. If B is an ideal in a ring R, the *integral closure* B_a of B is the set of elements x in R such that x satisfies an equation of the form $x^n+b_1x^{n-1}+\cdots+b_n=0$, where $b_i\in B^i$ $(i=1,\cdots,n)$. An ideal B in R is *semi-prime* in case B is an intersection of prime ideals. If R is an integral domain, then R is *normal* in case R is integrally closed in its quotient field. If R is a semi-local (Noetherian) ring, then R is *analytically unramified* in case the completion of R (with respect to the powers of the Jacobson radical of R) contains no nonzero nilpotent elements.

Let R be a semi-local ring with Jacobson radical J, and let R^* be the completion of R. In [2], Zariski proved that if R is a normal local integral domain, and if there is a nonzero element x in J such that $\mathfrak{p}R^*$ is semi-prime, for every prime divisor $\mathfrak{p}$ of xR, then R is analytically unramified. In [1, p. 132] Nagata proved that if R is a semi-local integral domain, and if there is a nonzero element x in J such that, for every prime divisor $\mathfrak{p}$ of xR, $\mathfrak{p}R^*$ is semi-prime and $R_{\mathfrak{p}}$ is a valuation ring, then R is analytically unramified. (The condition $R_{\mathfrak{p}}$ is a valuation ring holds if R is normal.) The main purpose of this note is to extend Nagata's result to the case where R is a semi-local ring (Theorem 1). This extension will be given after first proving a

Received by the editors March 29, 1965.

[1] Work on this paper was supported in part by the National Science Foundation. Grant GP3595.