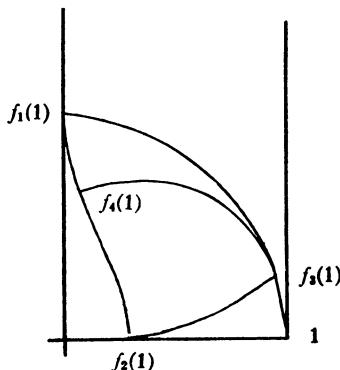


CONVERGENCE OF $i^{i^{\dots}}$

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Convergence of the sequence defined by $z_1 = i$, $z_{n+1} = \exp(i\pi z_n/2)$ follows from computations reported and discussed by D. L. Shell in his 1959 thesis. If we note that the function $w = \exp(i\pi z/2)$ maps the half strip $0 < R(z) < 1$, $I(z) > 0$ on to the quadrant $|w| < 1$, $0 < \arg w < \frac{1}{2}\pi$ the convergence is proved by considering iterations of this mapping. For there must exist a fixed point, or solution $z = \alpha$ of the equation $z = \exp(i\pi z/2)$ within the quadrant. From the work of Wolff and Denjoy α is unique and the sequence of iterates defined by $f_1(z) = \exp(i\pi z/2)$ and $f_{n+1}(z) = \exp(i\pi f_n(z)/2)$ converges uniformly to α for z in any closed set interior to the quadrant. Of course, in the special circumstances of the present example these results are elementary exercises. Since $f_4(1)$ is within the quadrant convergence of $i^{i^{\dots}}$ follows.



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