# UPPER BOUNDS FOR A BLOCH CONSTANT

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1. **Introduction.** It seems that the first numerical results concerning Bloch's theorem for an annulus arise from two examples discussed by Valiron [**1**]. He used a slightly different constant from the one we consider. It is, however, easy to modify his construction. In our notation Valiron's methods lead to the inequality $B(2) < 0.10025$. We give a very simple example to show that $B(2) < 0.0884$, and another depending on elliptic functions to show that $B(2) < 0.0746$.

DEFINITION. Let $w = f(z) = z + a_2z^2 + \cdots$ be regular in $|z| < 1$, then we define $B(2)$ as the supremum of all numbers $\lambda$ such that the Riemann surface of the inverse function $z = f^{-1}(w)$ contains a schlicht annulus of the form $\rho < |w| < 2\rho + \epsilon$ cut along a radius where $\epsilon > 0$, and $\rho \geqq \lambda$. For the existence of $B(2)$ see [**2**].

Valiron's first example which was due to H. Cartan is as follows: Consider the function

$$(1) \quad w = f(z) = \frac{U(z) - U(0)}{U'(0)}, \qquad U(z) = \frac{Z}{(Z-1)^2}, \qquad Z = e^{(z-1)/(z+1)}.$$

By elementary calculations, we find

$$(2) \qquad B(2) < 0.1156.$$

The second example is as follows:
Consider the function

$$(3) \quad w = f(z) = \frac{U(z) - U(0)}{U'(0)}, \quad U(z) = \frac{Z^a}{(Z^a - 1)^2}, \quad Z = e^{(z-1)/(z+1)}$$

for $|z| < 1$ where $a$ is a positive number.

In this example, $f(z)$ will omit the value

$$(4) \qquad \frac{-U(0)}{U'(0)} = \frac{-(e^a - 1)}{2a(e^a + 1)},$$

and all points of the negative axis from

$$(5) \qquad \frac{-1/4 - U(0)}{U'(0)} = \frac{-(e^a + 1)(e^a - 1)}{8ae^a}$$

to minus infinity.

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To obtain the best upper bound of $B(2)$ from this example, we must choose $a$ to satisfy

$$(6) \qquad e^a = 3 + \sqrt{8},$$

that is

$$(7) \qquad a \doteqdot 1.7627,$$

and have

$$(8) \qquad B(2) < 0.10025.$$

(We have deviated from Valiron only in our choice of $a$.)

These two examples of course also limit the constants obtained when the condition "schlicht" is dropped.

2. **New examples.** The following two examples will give us better estimates for $B(2)$.

EXAMPLE 1. Let us construct the function $w=f(z)=z+a_2z^2+\cdots$ which is regular in $|z|<1$ and maps $|z|<1$ on the Riemann surface with a winding point of order 2 at $w=\alpha^*$ and a cut from $w=2\alpha^*$ radially to infinity.

Consider the function $w$ of $z$ defined by

$$(1) \quad V = \frac{z+i\alpha}{1-i\alpha z}, \quad U = V^2, \quad W = \frac{U}{(1+U)^2}, \quad w = c\left(W + \frac{1}{4}\right).$$

Note that $|z|<1$ is represented in this way on a Riemann surface of two sheets over the $W$ plane which has $W=0$ as a winding point and the segments of the positive real axis from $1/4$ to infinity as cuts. The corresponding positions of winding point and end of cut in the $W+1/4$ plane will be $1/4$ and $1/2$. We also require $z=0$ to correspond to $w=0$.

Let us set $W=-1/4$ for $U=-\alpha^2$, then $U=-3\pm\sqrt{8}$ and $V=\pm i\alpha$. Hence

$$(2) \qquad \alpha = \sqrt{2} - 1.$$

It only remains to find the value of $c$ such that $f'(0)=1$, and then $\alpha^*$ will equal $c/4$. For this, since

$$(3) \qquad \frac{dw}{dz} = c\cdot\frac{1-U}{(1+U)^3}\cdot 2V\cdot\frac{1-\alpha^2}{(1-i\alpha z)^2} = c\cdot\frac{1-V^2}{(1+V^2)^3}\cdot 2V\cdot\frac{1-\alpha^2}{(1-i\alpha z)^2},$$

$$\left.\frac{dw}{dz}\right|_{z=0} = 1 \quad \text{implies} \quad c = \frac{(1-\alpha^2)^2}{2i\alpha(1+\alpha^2)} = -\frac{i}{\sqrt{2}}\,.$$

Thus the desired function is as follows;

$$(4)\qquad w = -\frac{i}{\sqrt{2}}\left(W + \frac{1}{4}\right), \qquad W = \frac{U}{(1+U)^2},$$

$$U = V^2, \qquad V = \frac{z + i(\sqrt{2}-1)}{1 - i(\sqrt{2}-1)z}.$$

Since $|c|/4 = \sqrt{2}/8 < 0.1768$, we have

$$(5)\qquad B(2) < 0.0884.$$

EXAMPLE 2. This example is in fact an extension of Example 1. Let us construct the function $w = f(z) = z + a_2 z^2 + \cdots$ which is regular in $|z| < 1$ and maps $|z| < 1$ on the Riemann surface with winding points of order 2 at $w = \alpha$ and at $w = 2\alpha$ and a cut from $w = 4\alpha$ radially to infinity.

By using a linear transformation between $|z| < 1$ and $|Z| < 1$ to map that point in $|z| < 1$ which is the corresponding point of the first double point into the origin of the $Z$-plane, we get the system of equivalent regions, $\cdots S^3, S^2, S^1, S_0, S_1, S_2, \cdots$ in $|Z| < 1$ (see Figure) where the upper and the lower indices indicate the corresponding sheets of the Riemann surface of the inverse function $z = f^{-1}(w)$ of the desired function. This system of regions, in fact, has infinitely many members related mutually by a transformation which is the composition of a pair of elliptic transformations,

$$(6)\qquad \zeta' = -\zeta$$

and

$$(7)\qquad \frac{\zeta' - i\delta}{1 + i\delta\zeta'} = -\frac{\zeta - i\delta}{1 + i\delta\zeta}$$

in the following way, for instance,

$$S_0 \xrightarrow{(6)} S_1 \xrightarrow{(7)} S^1 \xrightarrow{(6)} S_2 \xrightarrow{(7)} S^3 \longrightarrow$$

$$S_1 \xrightarrow{(6)} S_0 \xrightarrow{(7)} S^2 \xrightarrow{(6)} S_3 \xrightarrow{(7)} S^\sigma \xrightarrow{(7)} \text{ etc.}$$

By simple calculation, the composition (6) $\circ$ (7) of (6) and (7) will be a hyperbolic transformation

$$(8)\qquad \frac{\zeta^* - i}{\zeta^* + i} = \left(\frac{1-\delta}{1+\delta}\right)^2 \cdot \frac{\zeta - i}{\zeta + i}.$$

After this preparation, the desired function will be obtained by the successive composition of linear transformations, an exponential and an elliptic function starting from the upper-half plane of a sheet of the Riemann surface.

Let us first find the linear transformation between the upper-half plane of the $w$-plane and the upper-half plane of the $W$-plane such that $w=\alpha$, $w=2\alpha$, $w=4\alpha$, $w=\infty$ go to $W=-1/k$, $W=-1$, $W=1$, $W=1/k$ respectively. The required transformation is

$$W = \frac{\sqrt{3}w - (3+\sqrt{3})\alpha}{(2\sqrt{3}-3)w - (5\sqrt{3}-9)\alpha} \tag{9}$$

with $k=2-\sqrt{3}$. We also find that

$$W = W_0 = -(7+4\sqrt{3}) \doteqdot -13.92820, \tag{10}$$

when $w=0$.

Next we want to find the transformation between the upper-half plane of the $W$-plane and the interior of the rectangle of the $u$-plane such that $W=-1/k$, $W=-1$, $W=1$, $W=1/k$ go to $u=-K+iK'$, $u=-K$, $u=K$, $u=K+iK'$ respectively. This is given [**3**] by

$$u = \int_0^W \frac{dW}{((1-W^2)(1-k^2W^2))^{1/2}} \tag{11}$$

where

$$K = \int_0^1 \frac{dt}{((1-t^2)(1-k^2t^2))^{1/2}}, \qquad K' = \int_0^1 \frac{dt}{((1-t^2)(1-k'^2t^2))^{1/2}}$$

and $k'=(1-k^2)^{1/2}=(4\sqrt{3}-6)^{1/2}$. Using tables [**4**] we have $K\doteqdot 1.60047$ and $K'\doteqdot 2.73955$.

Let $u_0$ be the corresponding value of $W_0$, then since $sn(u_0-iK') = \{k\cdot sn(u_0)\}^{-1}$, by using mathematical tables again we have

$$u_0 \doteqdot -0.27148 + 2.73955i. \tag{12}$$

After (11) we continue to apply the following successive transformations:

$$U = u + K, \tag{13}$$

$$V = i\pi U/4K, \tag{14}$$

$$\zeta = \exp(V + \pi K'/4K), \tag{15}$$

$$Z = i\frac{\zeta - 1}{\zeta + 1}, \tag{16}$$

$$z = \frac{Z - Z_0}{1 - Z_0 Z}. \tag{17}$$

The values $U_0$, $V_0$, $\zeta_0$ and $Z_0$ which are the values taken at $z=0$ are

$$\begin{aligned} U_0 &\doteqdot 1.32899 + 2.73955i, \\ V_0 &\doteqdot -1.34438 + 0.65217i, \\ \zeta_0 &\doteqdot e^{0.65217i}, \\ Z_0 &\doteqdot -0.33815. \end{aligned} \tag{18}$$

Thus, we are almost in the position to construct the desired function $w=f(z)=z+a_2z^2+\cdots$. It only remains to find the value $\alpha$ such that $f'(0)=1$.

For this, since

$$\begin{aligned} \left.\frac{dw}{dz}\right|_{z=0} &= \frac{(4\sqrt{3}-6)\alpha}{\{(2-\sqrt{3})W_0 - 1\}^2}\cdot\left\{-((1-W_0^2)(1-k^2W_0^2))^{1/2}\right\} \\ &\quad \cdot\frac{4K}{\pi i}\cdot\frac{1}{\zeta_0}\cdot\frac{2i}{(i-Z_0)^2}\cdot(1-Z_0^2) = 1, \end{aligned} \tag{19}$$

$$\alpha = \frac{-(i-Z_0)^2\cdot\zeta_0\cdot\pi\cdot\{(2-\sqrt{3})W_0 - 1\}^2}{(1-Z_0^2)\cdot 8K\cdot((1-W_0^2)(1-k^2W_0^2))^{1/2}\cdot(4\sqrt{3}-6)}. \tag{20}$$

Hence by substituting $W_0$, $u_0$, $U_0$, $V_0$, $\zeta_0$ and $Z_0$ with their numerical values, we have

$$\alpha < 0.1492.$$

Therefore

$$B(2) < 0.0746.$$

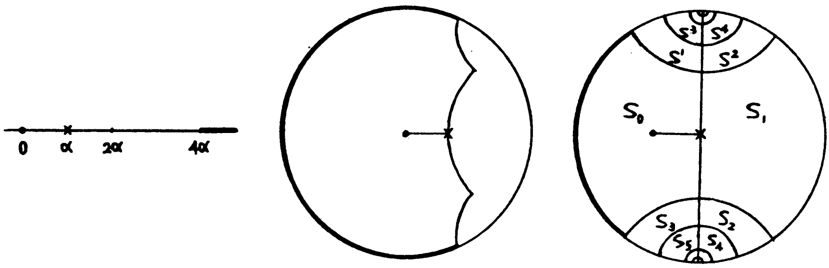
Remark. These new examples as well as Valiron's also limit an other kind of Bloch constant, say $B^*$, in which the schlicht circle: $|w - w_f| < P$ has its center and a whole diameter on the positive real axis with $P > B^*$. To find an upper bound of $B^*$, it would be better to set the end of the cut at $w=3\alpha$ instead of at $w=4\alpha$ in Example 2. In this case, the corresponding transformation of (9) will be

$$W = \frac{-\sqrt{8}w + (4+\sqrt{8})\alpha}{(8-3\sqrt{8})w + (7\sqrt{8}-20)\alpha} \tag{9'}$$

with $k=3-\sqrt{8}$. Hence subsequently $k'=(6\sqrt{8}-16)^{1/2}$, $K\doteqdot 1.582$ and $K'\doteqdot 3.169$.

With the other transformations (11), (13), (14), (15), (16), (17) remain unchanged, we can also calculate the values $W_0$, $u_0$, $U_0$, $V_0$, $\zeta_0$ and $Z_0$, and similarly can find that $\alpha$ is less than 0.16. Hence

$$B^* < 0.08. \tag{21}$$




W-PLANE Z-PLANE Ƶ-PLANE

## REFERENCES

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