SOME INEQUALITIES FOR HYPERGEOMETRIC FUNCTIONS¹

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1. Introduction. The hypergeometric R -function, $R(a; b_1, \dots, b_n; x_1, \dots, x_n)$, has recently been shown [4] to have a close connection with the theory of elementary mean values. By exploiting this connection we shall arrive at inequalities which compare R with (combinations of) mean values of the form $(\sum w_i x_i^{1/t})^{1/t}$, where the weights (w) are related to the b -parameters by $w_i = b_i / \sum b_i$, ($i = 1, \dots, n$). Both the b -parameters and the arguments (x) are required to be positive, but the a -parameter may be any real number.

The inequalities in their general form (Theorem 2) subsume a wide variety of special results, including some inequalities for elementary transcendental functions and elliptic integrals, inequalities due to Watson [10] and Szegő [9] for a certain integral, and new inequalities for the Gaussian and confluent hypergeometric functions as well as Appell's function F_1 . New inequalities for the surface area and capacity of an ellipsoid in n dimensions are also included.

This work began from a conversation with Dr. G. D. Chakerian at a time when I was privileged to use the facilities of the Mathematics Department of the California Institute of Technology. I wish to thank also Mr. Malcolm D. Tobey for discussions of Theorem 1, which he first proved for restricted values of t by a method different from the one used here.

2. The inequalities. The R -function will be defined for present purposes by an integral representation,

$$(2.1) \quad R(a, b, x) = \int_E \left(\sum_{i=1}^n u_i x_i \right)^{-a} P(b, u') du',$$

where (b) and (x) stand for n -tuples of positive numbers, $(u') = (u_1, \dots, u_{n-1})$, and $du' = du_1 \cdots du_{n-1}$. The domain of integration E is the set of points satisfying $u_i > 0$ ($i = 1, \dots, n-1$) and $\sum_{i=1}^{n-1} u_i < 1$. Because we define $u_n \equiv 1 - \sum_{i=1}^{n-1} u_i$, the last condition is equivalently $u_n > 0$. The positive weight function

Received by the editors April 13, 1965.

¹ Research supported by the National Aeronautics and Space Administration (under Grant NsG-293 to Iowa State University) and by the Ames Laboratory of the U. S. Atomic Energy Commission.

$$P(b, u') = \frac{\Gamma(b_1 + \cdots + b_n)}{\Gamma(b_1) \cdots \Gamma(b_n)} \prod_{i=1}^n u_i^{b_i-1}, \quad (u') \in E,$$

satisfies

$$\int_E P(b, u') du' = 1.$$

A homogeneous mean of the values (x) is constructed from the R -function in the following way. We define the parameter c , the positive weights (w) , and the hypergeometric mean $M(t, c)$ by

$$(2.2) \quad c \equiv \sum_{i=1}^n b_i,$$

$$(2.3) \quad w_i \equiv b_i/c, \quad (i = 1, \cdots, n), \quad \left(\sum_{i=1}^n w_i = 1 \right),$$

$$(2.4) \quad M(t, c; x, w) \equiv [R(-t, cw, x)]^{1/t}, \quad (t \neq 0, c > 0).$$

If $t=0$ or $c=0$, the mean value is defined to be the limiting value of $M(t, c)$ as $t \rightarrow 0$ or $c \rightarrow 0$. It is shown in [4] that

$$(2.5) \quad \lim_{c \rightarrow 0} M(t, c; x, w) = M_t(x, w),$$

where M_t is the mean of order t :

$$(2.6) \quad M_t(x, w) \equiv \left(\sum_{i=1}^n w_i x_i^t \right)^{1/t}, \quad (t \neq 0),$$

$$M_0(x, w) \equiv \prod_{i=1}^n x_i^{w_i}.$$

We shall want several results taken from [4] which are valid for $c \geq 0$:

$$(2.7) \quad \begin{aligned} M(s, c; x, w) &< M(t, c; x, w) \quad \text{if } s < t \text{ and } \min(x) < \max(x); \\ M(1, c; x, w) &= M_1(x, w), \quad M(-c, c; x, w) = M_0(x, w). \end{aligned}$$

By $\max(x)$ we denote the largest of the positive numbers $x_1, \cdots, x_n$. In addition to (2.7) we shall want a new result:

THEOREM 1. *If $\min(x) < \max(x)$, the hypergeometric mean of the positive values (x) with positive weights (w) satisfies*

$$(2.8) \quad \begin{aligned} M(t, c; x, w) &< M_t(x, w), \quad (t > 1, c > 0), \\ M(t, c; x, w) &> M_t(x, w), \quad (t < 1, c > 0). \end{aligned}$$

PROOF. By (2.1) and (2.4) we have

$$(2.9) \quad [M(t, c; x, w)]^t = \int_E \left(\sum_{i=1}^n u_i x_i \right)^t P(cw, u') du', \quad (c > 0).$$

If $\min(x) < \max(x)$, then $(\sum u_i x_i)^t < \sum u_i x_i^t$ for $t > 1$ or $t < 0$, with reversed inequality for $0 < t < 1$; this follows directly from the fact that $M_t(x, u)$ is a strictly increasing function of t . Using (2.9) and (2.7), we have

$$\begin{aligned} \int_E (\sum u_i x_i^t) P(cw, u') du' &= M(1, c; x^t, w) = M_1(x^t, w) \\ &= \sum w_i x_i^t = [M_t(x, w)]^t. \end{aligned}$$

The inequalities of Theorem 1 follow immediately provided that $t \neq 0$. If $t = 0$ and $c > 0$, the integral representation [4] of $M(0, c)$ and the concavity of the logarithmic function show that

$$\begin{aligned} \log[M(0, c; x, w)] &= \int_E \log \left(\sum_{i=1}^n u_i x_i \right) P(cw, u') du' \\ (2.10) \quad &> \int_E \left(\sum_{i=1}^n u_i \log x_i \right) P(cw, u') du' \\ &= \sum w_i \log x_i = \log M_0(x, w). \end{aligned}$$

In preparation for the statement of Theorem 2, we define several quantities in terms of elementary mean values, assuming always that (b) , c , and (w) are related by (2.2) and (2.3):

$$\begin{aligned} H(a, b, x) &\equiv [M_1(x, w)]^{-a} = (\sum b_i x_i / c)^{-a}, \\ J(a, b, x) &\equiv [M_{-a}(x, w)]^{-a} = \sum b_i x_i^{-a} / c, \\ K(a, b, x) &\equiv [M_0(x, w)]^{-a} = \prod x_i^{-ab_i/c}, \\ (2.11) \quad J'(a, b, x) &\equiv [M_0(x, w)]^{-c} [M_{c-a}(x, w)]^{c-a} = (\prod x_i^{-b_i}) \sum b_i x_i^{c-a} / c, \\ H'(a, b, x) &\equiv [M_0(x, w)]^{-c} [M_{-1}(x, w)]^{c-a} \\ &= (\prod x_i^{-b_i}) (\sum b_i x_i^{-1} / c)^{a-c}. \end{aligned}$$

As implied by (2.7) and (2.20) below, there are several values of a for which the hypergeometric function $R(a, b, x)$ takes a simple form:

$$\begin{aligned} R &= M_1 = H = J, & (a = -1), \\ (2.12) \quad R &= 1 = H = J = K, & (a = 0), \\ R &= M_0^{-c} = H' = J' = K, & (a = c), \\ R &= M_0^{-c} M_{-1}^{-1} = H' = J', & (a = c + 1). \end{aligned}$$

Having disposed of these special values, we now state

THEOREM 2. *If the arguments (x) and parameters (b) are all positive and if $\min(x) < \max(x)$, the hypergeometric function $R(a, b, x)$ and the elementary functions defined by (2.11) satisfy*

$$(2.13) \quad H' < K < H < R < J < J', \quad (-\infty < a < -1),$$

$$(2.14) \quad H' < K < J < R < \min\{H, J'\}, \quad (-1 < a < 0),$$

$$(2.15) \quad \max\{H, H'\} < R < K < \min\{J, J'\}, \quad (0 < a < c),$$

$$(2.16) \quad H < K < J' < R < \min\{H', J\}, \quad (c < a < c+1),$$

$$(2.17) \quad H < K < H' < R < J' < J, \quad (c+1 < a < \infty).$$

If $a \leq c-1$, then $\min\{H, J'\} = H < J'$. If $a \geq 1$, then $\min\{H', J\} = H' < J$.

PROOF. The main inequalities are those relating to R ; all other parts of the theorem follow easily from the inequality $M_s < M_t$ if $s < t$. It is evident from (2.7) and (2.8) that

$$(2.18) \quad \begin{aligned} M_1 &< M(t, c) < M_t, & (1 < t < \infty), \\ M_t &< M(t, c) < M_1, & (0 < t < 1), \\ M_0 &< M(t, c) < M_1, & (-c < t < 0), \\ M_t &< M(t, c), & (-c-1 < t < -c). \end{aligned}$$

Several additions to this list could be made (for instance, the last inequality is valid also for $-\infty < t < -c-1$) without sharpening the final results. We now raise each of the inequalities (2.18) to the power t , use (2.4), and substitute $t = -a$, thereby proving those parts of Theorem 2 which compare R with H , J , or K . In the inequalities

$$H(a, b, x) < R(a, b, x) < J(a, b, x), \quad (-\infty < a < -1),$$

we now replace a by $c-a$ and (x) by $(x^{-1}) \equiv (x_1^{-1}, \dots, x_n^{-1})$. The elementary relation $M_t(x^{-1}, w) = [M_{-t}(x, w)]^{-1}$ implies that

$$(2.19) \quad H(c-a, b, x^{-1}) = [M_0(x, w)]^c H'(a, b, x)$$

and that (2.19) still holds if H and H' are replaced by J and J' (or by K and $K' = K$). The Euler transformation of R is a similar equation [2]:

$$(2.20) \quad R(c-a, b, x^{-1}) = \left(\prod_{i=1}^n x_i^{b_i} \right) R(a, b, x) = [M_0(x, w)]^c R(a, b, x).$$

Thus (2.13) implies (2.17), and similarly all those parts of Theorem 2

which compare R with H' or J' are implied by those parts which compare R with H or J .

3. **Some special cases.** Assume x and y to be positive and unequal. From (2.1) and (2.15) we find that

$$R(1; 1, 1; x, y) = (\log x - \log y)/(x - y)$$

satisfies $H < R < K$; hence

$$(3.1) \quad (xy)^{1/2} < \frac{x - y}{\log x - \log y} < \frac{x + y}{2}.$$

If $r \neq 0$, then $R(1 - r; 1, 1; x, y) = (x^r - y^r)/r(x - y)$ lies always between $H = [\frac{1}{2}(x + y)]^{r-1}$ and $J = \frac{1}{2}(x^{r-1} + y^{r-1})$ according to Theorem 2. This statement is sharper than the very useful result [5, Theorem 41] that it lies always between x^{r-1} and y^{r-1} .

If $0 < a < c$ the R -function has the integral representation [2, Equation (7.1)]

$$(3.2) \quad B(a, c - a)R(a, b, x) = \int_0^\infty t^{c-a-1} \prod_{i=1}^n (t + x_i)^{-b_i} dt,$$

where B is the beta function. The integral was considered by Watson [10] and Szegő [9], who proved the right and left parts, respectively, of the inequality $H < R < K$ for the case $0 < a = c - 1$. This is a special case of (2.15).

Gauss' hypergeometric function is expressible in terms of the R -function by

$$(3.3) \quad {}_2F_1(a, b; c; x) = R(a; b, c - b; 1 - x, 1).$$

Assume $c > b > 0$ and $-\infty < x < 1$, $x \neq 0$. Then the inequalities of Theorem 2 remain valid when the following substitutions are made:

$$(3.4) \quad \begin{aligned} R &= {}_2F_1(a, b; c; x), \\ K &= (1 - x)^{-ab/c}, \\ H &= (1 - bx/c)^{-a}, \\ H' &= (1 - x)^{c-a-b} [1 - x + bx/c]^{a-c}, \\ J &= 1 + (b/c)[(1 - x)^{-a} - 1], \\ J' &= (b/c)(1 - x)^{c-a-b} + (1 - b/c)(1 - x)^{-b}. \end{aligned}$$

If $c > a > 0$ it may be useful to recall that both members of (3.3) are symmetric in a and b .

Inequalities for confluent hypergeometric functions can be obtained by taking a limit of Equations (3.4) in either of two ways. In the first limiting process (replacing x by x/a and letting $a \rightarrow \infty$), (2.13) or (2.17) yields only a weaker form (with $<$ replaced by $\leq$) of the inequalities

$$(3.5) \quad e^{bx/c} < {}_1F_1(b; c; x) < 1 + (b/c)(e^x - 1), \quad (c > b > 0, x \neq 0).$$

However, strict inequality is easy to prove if $x > 0$ by comparing the power series of the three members, and if $x < 0$ by Kummer's transformation of ${}_1F_1$. In the other limiting process (replacing x by cx and letting $c \rightarrow \infty$), (2.13)–(2.15) become the following inequalities, valid for $b \geq 0$ and $x \leq 0$:

$$(3.6) \quad (1 - bx)^{-a} \leq {}_2F_0(a, b; x), \quad (-\infty < a \leq -1),$$

$$(3.7) \quad 1 \leq {}_2F_0(a, b; x) \leq (1 - bx)^{-a}, \quad (-1 \leq a \leq 0),$$

$$(3.8) \quad (1 - bx)^{-a} \leq {}_2F_0(a, b; x) \leq 1, \quad (0 \leq a < \infty).$$

Results analogous to those for Gauss' hypergeometric function are easily found for Appell's function F_1 by using the relation

$$(3.9) \quad F_1(a, b, b'; c; x, y) = R(a; b, b', c - b - b'; 1 - x, 1 - y, 1).$$

4. Area and capacity of an ellipsoid. The ellipsoid $\sum_{i=1}^n x_i^2/a_i^2 = 1$ has volume $V = a_1 a_2 \cdots a_n \pi^{n/2} / \Gamma(\frac{1}{2}n + 1)$; its surface area S and capacity C are given by

$$(4.1) \quad S/nV = R(-\tfrac{1}{2}; \tfrac{1}{2}, \dots, \tfrac{1}{2}; \bar{a}_1^{-2}, \dots, \bar{a}_n^{-2}), \quad (n \geq 2),$$

$$(4.2) \quad (n-2)/C = R(\tfrac{1}{2}n-1; \tfrac{1}{2}, \dots, \tfrac{1}{2}; a_1^2, \dots, a_n^2), \quad (n \geq 3).$$

The expression for S is derived by writing the surface area as an $(n-1)$ -fold integral [6, Equation (10)] and comparing with (2.1); the expression for C comes from separating Laplace's equation in confocal coordinates [1] and using (3.2).

Assume that $a_1, \dots, a_n$ are positive, finite, and not all equal. Then Theorem 2 gives inequalities for S and C , in particular

$$(4.3) \quad \prod a_i^{-1/n} < \sum a_i^{-1/n} < S/nV < (\sum a_i^{-2/n})^{1/2}.$$

For $n=3$ these inequalities were found by Peano and Pólya [7]; for general n the members of (4.3) were discussed by Lehmer [6, p. 230] as approximations (but not as bounds) for S in the case of nearly spherical ellipsoids.

If $n=3$, inequalities for C and S are equivalent to inequalities for the symmetric normal elliptic integrals [3]

$$(4.4) \quad \begin{aligned} R_F(x, y, z) &\equiv R(\tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}; x, y, z), \\ R_G(x, y, z) &\equiv R(-\tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}; x, y, z). \end{aligned}$$

If x, y, z are positive and not all equal, Theorem 2 shows that

$$(4.5) \quad \left(\frac{3}{x+y+z} \right)^{1/2} < R_F(x, y, z) < (xyz)^{-1/6},$$

$$(4.6) \quad (xyz)^{1/6} < \frac{x^{1/2} + y^{1/2} + z^{1/2}}{3} < R_G(x, y, z) < \left(\frac{x+y+z}{3} \right)^{1/2}.$$

Inequalities for C due to Pólya and Szegő [8, Equation (1.7)] are equivalent to inequalities for R_F that are sharper than (4.5):

$$(4.7) \quad \frac{3}{x^{1/2} + y^{1/2} + z^{1/2}} < R_F(x, y, z) < \frac{3}{(yz)^{1/4} + (zx)^{1/4} + (xy)^{1/4}}.$$

The left side of (4.7) is generalized to any $n \geq 3$ by

THEOREM 3. *If $x_1, \dots, x_n$ are nonnegative and not all equal, then*

$$(4.8) \quad \left(n / \sum_{i=1}^n x_i^{1/2} \right)^{n-2} < R(\tfrac{1}{2}n - 1; \tfrac{1}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2}; x_1, x_2, \dots, x_n),$$

($n \geq 3$).

PROOF. Assume $y_1, \dots, y_n$ are positive and not all equal. The inequality

$$(\sum y_i/n)^{n-1} > (\prod y_i) \sum y_i^{-1}/n, \quad (n \geq 3),$$

is part of Maclaurin's theorem for elementary symmetric functions [5, Theorem 52]. Replacing y_i by $(t+x_i)^{1/2}$, where $t > 0$, we have

$$\begin{aligned} \prod (t+x_i)^{-1/2} &> [\sum (t+x_i)^{1/2}/n]^{1-n} \sum (t+x_i)^{-1/2}/n \\ &= (1 - \tfrac{1}{2}n)^{-1} (d/dt) [\sum (t+x_i)^{1/2}/n]^{2-n}. \end{aligned}$$

We now integrate with respect to t and use (3.2) with $a = \frac{1}{2}n - 1 = c - 1$.

Theorems 2 and 3 show that the capacity of an ellipsoid in n dimensions ($n \geq 3$) with positive semiaxes satisfies

$$(4.9) \quad \prod a_i^{1-2/n} < C/(n-2) < (\sum a_i/n)^{n-2}$$

unless the ellipsoid is a sphere.

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