SUCCESSIVE DIFFERENCES OF BOUNDED SEQUENCES

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Let $x_0, x_1, x_2, \cdots$ be a bounded sequence and write $\Delta x_0 = x_1 - x_0$, $\Delta^2 x_0 = x_2 - 2x_1 + x_0, \cdots$ and in general

$$\Delta^n x_0 = \sum_{k=0}^{n} (-1)^k \binom{n}{k} x_{n-k}.$$

Of course if $\Delta^n x_0 \equiv 0$ from some point on then $\{x_n\}$ is a constant sequence. On the other hand $\Delta^n x_0$ can approach 0 very rapidly without the sequence being a constant, e.g. if $x_n = 1/2^n$ then $\Delta^n x_0 = (-\frac{1}{2})^n$. We will determine the exact rapidity with which $\Delta^n x_0$ can go to 0 for a nonconstant sequence.

THEOREM. A. *Let $c > 0$. There exists a nonconstant bounded sequence $\{x_n\}$ with $|\Delta^n x_0| \leqq (c/n)^n$.*

B. *Let $\{x_n\}$ be a bounded sequence for which $n|\Delta^n x_0|^{1/n} \to 0$. Then $\{x_n\}$ is a constant.*

PROOF. A. Choose

$$x_n = \sum_{k=0}^{n} \frac{(-\delta)^k}{k!} \binom{n}{k}$$

so that $\Delta^n x_0 = (-\delta)^n/n!$ and $|\Delta^n x_0| \leqq (c/n)^n$ when $\delta = c/e$. This sequence is certainly nonconstant and we need only show that it is bounded. We have

$$\sum_{k=0}^{n} \frac{(-\delta)^k}{k!} \binom{n}{k} = 1/2\pi i \int_C e^{-\delta z}(1 + 1/z)^n \, dz/z \tag{1}$$

where C is the circle $|z| = (n/\delta)^{1/2}$.

Write $z = (n/\delta)^{1/2} e^{i\theta}$ and observe that

$$\begin{aligned} e^{-\delta/n} &\left| e^{-\delta z/n}(1 + 1/z) \right|^2 \\ &= e^{-\delta/n} \exp\left[-2(\delta/n)^{1/2} \cos\theta\right](1 + 2(\delta/n)^{1/2} \cos\theta + \delta/n) \\ &\leqq 1, \end{aligned}$$

since $e^{-t}(1 + t) \leqq 1$ for all real t. It follows immediately that $|e^{-\delta z}(1+1/z)^n| \leqq \exp[\delta/n \cdot n/2] = e^{\delta/2}$ and so by (1) we obtain the bound $|x_n| \leqq e^{\delta/2}$.

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B. Write $F(z)=e^{-z}\sum x_n(z^n/n!)$ and note that along the positive axis

$$(2) \qquad |F(z)| \leqq e^{-z}\sum |x_n| \frac{z^n}{n!} \leqq e^{-z}Ce^{z} = C.$$

Also

$$F(z) = \sum \Delta^n x_0 \frac{z^n}{n!}$$

and by hypothesis for every $\epsilon>0$ a C_ϵ can be found so that $|\Delta^n x_0| \leqq C_\epsilon(\epsilon^2/2n)^n$. Hence, for all z,

$$|F(z)| \leqq C_\epsilon \sum \left(\frac{\epsilon^2}{2n}\right)^n \frac{|z|^n}{n!} \leqq C_\epsilon \sum \frac{(\epsilon|z|^{1/2})^{2n}}{(2n)^n n!}$$
$$\leqq C_\epsilon \sum \frac{(\epsilon|z|^{1/2})^{2n}}{2n!}$$

so that

$$(3) \qquad |F(z)| \leqq C_\epsilon \exp[\epsilon|z|^{1/2}].$$

The proof is now completed by an application of the Phrágmen-Lindelöff theorems.

We quote Titchmarch **[1]**:

Let $f(z)$ be analytic in $|\arg z| < \pi/2\alpha$, continuous in $|\arg z| \leqq \pi/2\alpha$. Suppose that $|f(z)| \leqq C$ on $|\arg z| = \pi/2\alpha$ and that $|f(z)| \leqq C_\epsilon \exp[\epsilon|z|^\alpha]$ for every $\epsilon>0$ in $|\arg z| < \pi/2\alpha$. Then $|f(z)| \leqq C$ throughout $|\arg z| \leqq \pi/2\alpha$.

Applying this to $f(z)=F(-z)$, $\alpha=\frac{1}{2}$ we conclude by (2) and (3) that $|F(-z)| \leqq C$ throughout the plane so that, since F is entire, it is a constant. Thus $e^{-z}\sum x_n(z^n/n!)=C$ and so $x_n=C$.

Reference

1. E. C. Titchmarch, *The theory of functions*, 2nd ed., Oxford Univ. Press, New York, 1961; p. 178.

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