INCLUSION RELATIONS AMONG ORLICZ SPACES

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This paper contains two results; the first extends to a wide class of Orlicz spaces the statement, due to Krasnosel'skii and Rutickii [**1**, p. 60], that L_1 is the union of the Orlicz spaces which it contains properly; the second shows that for a wide class of spaces this is not true, i.e. there exists a set of Orlicz spaces no one of which is the union of the Orlicz spaces it contains properly. Here the Orlicz spaces are defined on $[0, 1]$ which is given Lebesgue measure μ.

1. We give in this section several definitions together with some elementary results about Orlicz spaces and convex functions.

Let $\mathcal{C}$ be the set of convex symmetric functions $\Phi: (-\infty, \infty) \to [0, \infty)$ such that $\Phi(0) = 0$, $\lim_{s\to 0} \Phi(s)/s = 0$ and $\lim_{s\to\infty} \Phi(s) = \infty$. If Φ and Ω are two elements of $\mathcal{C}$, we say $\Phi \leqq \Omega$ if there exist constants c and s_0 such that $\Phi(s) \leqq \Omega(cs)$ for all $s \geqq s_0$. We say $\Phi \sim \Omega$ if $\Phi \leqq \Omega$ and $\Omega \leqq \Phi$; we say $\Phi < \Omega$ if $\Phi \leqq \Omega$ but $\Omega \not\leqq \Phi$. If $\Phi_1 \sim \Phi_2$ and $\Phi_1 \leqq \Omega_1$ ($\Phi_1 < \Omega_1$) then $\Phi_2 \leqq \Omega_2$ ($\Phi_2 < \Omega_2$).

If $\Phi \in \mathcal{C}$, then there exists a nondecreasing function $\phi: [0, \infty) \to [0, \infty)$ such that $\phi(0) = 0$, $\lim_{s\to\infty} \phi(s) = \infty$ and

$$\Phi(s) = \int_0^{|s|} \phi(t)\, dt$$

(see [**1**, p. 5]). This representation for Φ yields easily the two following inequalities:

$$\frac{s}{2}\,\phi\left(\frac{s}{2}\right) \leqq \Phi(s) \leqq \phi(s), \tag{1}$$

$$2\Phi(s) \leqq \Phi(2s). \tag{2}$$

Let

$$L_\Phi^* = (f \in L_1 : \Phi(cf) \in L_1 \text{ for some positive real number } c).$$

The set L_Φ^* is called an Orlicz space. It has a unique uniformity which is compatible with the order relation. Since this uniformity does not intervene in what follows, we do not give its definition; for this see [**1**, p. 69].

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The order relations among the elements of $\mathcal{C}$ give rise to order relations among the L_Φ^* as follows:

(a) $\Phi \leqq \Omega$ implies $L_\Omega^* \subset L_\Phi^*$.

(b) $\Phi \sim \Omega$ implies $L_\Omega^* = L_\Phi^*$.

(c) $\Phi < \Omega$ implies $L_\Omega^* \subset L_\Phi^*$ but $L_\Phi^* \not\subset L_\Omega^*$.

Statements a and b are direct consequences of the definitions; while c is a special case of

(d) $\lim \sup_{s\to\infty} \Omega(\alpha s)/\Phi(s) = \infty$ for all $\alpha > 0$ implies there exists $f \in L_\Phi$ such that $f \notin L_\Omega^*$.

Proof. Let $\overline{E}_{ij}$ be a pairwise disjoint double sequence of intervals in $[0, 1]$ such that $\mu(\overline{E}_{ij}) \neq 0$, $i, j = 1, 2, \cdots$. For each pair of natural numbers (n, i) there exists a number $s_{ni} > 0$ such that $s > s_{ni}$ implies $\Phi(s)\mu(\overline{E}_{ni}) > 2^{-i}n^{-2}$. There exist numbers $c_{ni} > s_{ni}$ such that $\Omega(c_{ni})/n > n^2\Phi(c_{ni})$.

Let E_{ni} be a nonempty subinterval of $\overline{E}_{ni}$ such that $\Phi(c_{ni})\mu(E_{ni}) = 2^{-i}n^{-2}$ and define

$$f(x) = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} c_{ij} X_{E_{ij}}(x).$$

It is easy to show that $f \in L_\Phi^*$ but $f \notin L_\Omega^*$.

(e) One can use a, b and d to show that $\Phi < \Omega$ if L_Ω^* is a proper subset of L_Φ^*.

2. We say that $\Phi \in \mathcal{C}$ satisfies * if

$$* \qquad \limsup_{s\to\infty} \frac{\Phi(2s)}{\Phi(s)} < \infty$$

and we say it satisfies ** if

$$** \qquad \liminf_{s\to\infty} \frac{\Phi(2s)}{s\Phi(s)} > 0.$$

These conditions are similar to the Δ_2 and Δ_3 conditioned in [1]. A function Φ which satisfies * grows less rapidly than some power and in addition it grows regularly; while a function Φ which satisfies ** grows like exp. It follows that $\mathcal{C}$ contains functions which satisfy neither * nor **.

Theorem 1. *Suppose $\Phi \in \mathcal{C}$ and Φ satisfies *; then, L_Φ^* is the union of the Orlicz spaces it contains properly.*

Proof. Let $f \in L_\Phi^*$. We will prove there exists Ω in $\mathcal{C}$ such that $\Phi < \Omega$ and such that $f \in L_\Omega^*$. If $f \in L_\infty$, we are finished because $L_\infty \subset L_\Phi^*$ properly. We will assume that $f(x) \geqq 0$ a.e.; this is in order because

$f \in L_\Phi^*$ iff $|f| \in L_\Phi^*$. Let c be a positive real number such that $\Phi(cf) \in L_1$. Define $\nu(s) = \mu((x : \Phi(cf(x)) \leqq s))$ and recall that

$$\int_0^1 u(\Phi(cf))\, d\mu = \int_0^\infty u(s)\, d\nu(s)$$

whenever u is integrable with respect to $d\nu$. The $sd\nu(s)$ measure of $[0, \infty)$ is finite (let $u(s) = s$) so there exists a function ω: $[0, \infty) \to [0, \infty)$ such that

(a′) $\int_0^\infty s\omega(s) d\nu(s) < \infty$,

(b′) ω is nondecreasing,

(c′) $\omega(0) = 0$ and $\lim_{s \to \infty} \omega(s) = \infty$.

The function

$$\Omega_0(s) = \int_0^{|s|} \omega(|s|)\, ds$$

is an element of $\mathcal{C}$ and so is $\Omega(s) = \Omega_0(\Phi(s))$ [1, p. 10]. The inequality 1 of §1 gives

$$\int_0^1 \Omega(cf)\, d\mu = \int_0^\infty \Omega_0(s)\, d\nu(s) \leqq \int_0^\infty s\omega(s)\, ds < \infty$$

from which it follows that $f \in L_\Omega^*$. To complete the proof, we must show $\Omega > \Phi$. Using inequality 1 again we get

$$\Omega(s) = \Omega_0(\Phi(s)) \geqq [\Phi(s)/2]\omega(\Phi(s)/2); \tag{3}$$

together with 2 this gives $\Omega(2s) \geqq \Phi(s)$ whenever $s \geqq s_0$. Here s_0 is any positive number such that $\omega(\Phi(s_0)/2) \geqq 1$. This shows that $\Omega \geqq \Phi$.

Let α be any positive number. There exist positive numbers s_0 and $M(\alpha)$ such that $\Phi(\alpha s) \geqq M(\alpha)\Phi(s)$ if $s \geqq s_0$; this is true because Φ satisfies *. Now this with 3 gives $\Omega(\alpha s) \geqq M(\alpha)\Phi(s)\omega(\Phi(\alpha s)/2)/2$ for $s \geqq s_0$ and this in turn gives

$$\limsup_{s \to \infty} \frac{\Omega(\alpha s)}{\Phi(s)} = \limsup_{s \to \infty} \frac{M(\alpha)\omega(\Phi(\alpha s)/2)}{2} = \infty.$$

Because α was arbitrary, we have that $\Omega \not\leqq \Phi$.

LEMMA. *Suppose Ω and Φ are two elements of $\mathcal{C}$ such that for some $\alpha > 1$*

$$\limsup_{s \to \infty} \frac{\Omega(s)}{\Phi(\alpha s)} \geqq 1.$$

Then, if t_0 is any positive number there exists $t \geqq t_0$ such that $\Omega(s) \geqq \Phi(s)$ for all $s \in [t, \alpha t]$.

PROOF. Let t_0 be any positive number and let $t \geqq t_0$ be any number such that $\Omega(t) \geqq \Phi(\alpha t)$. Let l' be a straight line through $(t, \Omega(t))$ which lies beneath the graph of Ω; such a line exists because Ω is convex (l' is not necessarily unique). Let l be the straight line, parallel to l' which passes through $(t, \Phi(\alpha t))$. Let u, v be the two numbers such that $u < v$ and l passes through the points $(u, \Phi(u))$, $(v, \Phi(v))$. By comparing similar triangles we get

$$\frac{\Phi(v) - \Phi(u)}{v - u} = \frac{\Phi(\alpha t) - \Phi(u)}{t - u}.$$

This leads directly to the inequality $v > \alpha t$. For $s \in [t, \alpha t]$, $(s, \Phi(s))$ is beneath the line l while $(s, \Omega(s))$ is above the line l'. Hence $\Phi(s) \leqq \Omega(s)$ for $s \in [t, \alpha t]$.

THEOREM 2. *Suppose $\Phi \in \mathcal{C}$ and satisfies **; then L_Φ^* is not the union of the Orlicz spaces it contains properly.*

PROOF. The condition ** implies there exists $\alpha > 0$ and s_0 such that $s \geqq s_0$ implies $\Phi(2s) > s\alpha\Phi(s)$. Let $c_n = 2^n s_0$. Let (E_n) be a sequence of pairwise disjoint subintervals of $[0, 1]$ such that $\Phi(c_n)\mu(E_n) = 2^{-n}$. This is possible because

$$\Phi(c_n) \geqq 2^{n(n-1)/2} s_0^{n-1} \alpha^n \Phi(s_0).$$

If we set

$$f(x) = \sum_{n=1}^{\infty} c_n X_{E_n}(x)$$

then $\Phi(f) \in L_1$ while $\Phi(2f) \notin L_1$; in fact

$$\int_{E_n} \Phi(2f)\, d\mu \geqq 2^n s_0 \alpha \Phi(c_n)\mu(E_n) = \alpha s_0.$$

Suppose L_Ω^* is a proper subset of L_Φ^*. This is the case only if $\Phi < \Omega$. Let k be any number in $(0, 1)$; we will show that $\Omega(kf) \notin L_1$ and hence that $f \notin L_\Omega$. The assertion $\Omega > \Phi$ implies that

$$\limsup_{s \to \infty} \frac{\Omega(ks)}{\Phi(2s)} = \infty.$$

Let $t_0 = c_1$ and apply the lemma to find t_1 such that $s \in [t_1, 2t_1]$ implies $\Omega(ks) \geqq \Phi(2s)$. Having chosen t_{n-1} choose $t_n > 2t_{n-1}$ such that $s \in [t_n, t_{2n}]$ implies $\Omega(ks) \geqq \Phi(2s)$. By induction this yields an infinite sequence of intervals $[t_n, 2t_n]$ each of which contains one of the numbers c_{m_n}.

The proof is completed by observing that

$$\int_0^1 \Omega(kf)\,d\mu \geqq \sum \Phi(2c_{m_n})\mu(E_{m_n}) \geqq \sum_{n=1}^{\infty} \alpha s_0.$$

Reference

1. Krasnosel'skii and Rutickii, *Convex functions and Orlicz spaces*, P. Noordhoff Ltd., Groningen, 1961.

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ON A COMBINATORIAL PROBLEM OF ERDÖS

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Let $C(n, m)$ denote the binomial coefficient $n!/(m!n-m!)$. Let S be a set containing N elements and let X be a collection of subsets of S with the property that if A, B and C are distinct elements of X, then $A\cup B\neq C$. Erdös [1], [2], has conjectured that X contains at most $KC(N, [N/2])$ elements where K is a constant independent of X and N. The problem is related to a result of Sperner [3] to the effect that if the collection X has the more restrictive property that no element of X contains any other, then X can have at most $C(N, [N/2])$ elements.

We show below that Erdös' conjecture for $K=2^{3/2}$ can be deduced directly from Sperner's result.

Let L_N be defined by

$$L_N \equiv 2^{[N/2]}C(N-[N/2], [\tfrac{1}{2}(N-[N/2])]) + 2^{N-[N/2]}C([N/2], [N/4]).$$

An easy calculation shows that L_N is always less than $2^{3/2}C(N, [N/2])$ to which it is asymptotic for large N. We prove:

Theorem. *If X is a family of subsets of an N element set S such that no three distinct A, B, C in X satisfy $A\cup B=C$, then X has less than L_N elements.*

Proof. For any finite set T and family X of subsets of T define

$$m_T(X) \equiv \{A\in X \mid B\in X \text{ and } B\subset A \text{ imply } B=A\}.$$

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