ON PURE STATES OF C^*-ALGEBRAS

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In the present note, we are concerned with the following problem (cf. [1, p. 58, problème]).

Let A be a C^*-algebra with unit 1, B a C^*-subalgebra of A containing 1, $\kappa(>0)$ a positive element of A. Then, does there exist a state f on A such that $f(\kappa)>0$ and f is a pure state on B?

We shall show that the answer for this problem is, in general, negative and give a sufficient condition in order that it is affirmative.

First of all we show

THEOREM 1. *Let A be a C^*-algebra, B a C^*-subalgebra of A, κ a positive element of A. Suppose that $f(\kappa)=0$ for all states f of A such that f is pure on B, then the C^*-subalgebra C generated by B and κ can be expressed as follows:*

$C=B+\Pi$, where Π is a closed self-adjoint, two-sided ideal of C, $B\cap\Pi=(0)$ and $\kappa\in\Pi$.

PROOF. Let T be the set of all states f of A such that f is pure on B and put $Q=\{y|f(y)=0$ for $y\in A$ and $f\in T\}$. Then for $b\in B$ and $y\in Q$, $f(b^*yb)=0$, because a state $g/f(b^*b)$ $(g(a)=f(b^*ab))$ is also pure on B if $f(b^*b)\neq 0$. Therefore $f(b^*\kappa b)=0$ for all $b\in B$ and all $f\in T$.

Let Π be a C^*-subalgebra generated by $\{\kappa, b^*\kappa c|b, c\in B\}$, then $|f(b_1^*\kappa c_1 b_2^*\kappa c_2\cdots b_n^*\kappa c_n)|\leqq f(b_1^*\kappa b_1)^{1/2}f(d)^{1/2}=0$, where d is some element of A.

Hence we can easily conclude $f(\Pi)=0$ for all $f\in T$; clearly $B\Pi\subset\Pi$ and $\Pi B\subset\Pi$, so that Π is an ideal of the C^*-algebra $B+\Pi$.

For $y\in B\cap\Pi$, $y^*y\in B\cap\Pi$ and so $f(y^*y)=0$ for all $f\in T$; hence $y^*y=y=0$. This completes the proof.

Now we shall show a sufficient condition.

COROLLARY. *Let A be a C^*-algebra, B a C^*-subalgebra of A, κ a positive element of A. Suppose that there is an element b in B such that $\|b-\kappa\|<\|b\|$, then there is a state f of A such that $f(\kappa)>0$ and f is pure on B.*

PROOF. Suppose that the statement is not true, then by Theorem 1 there is a closed ideal Π such that $B\cap\Pi=(0)$ and $\kappa\in\Pi$. The mapping $B\to B+\Pi/\Pi$ is isometric; hence $\|y+\kappa\|\geqq\|y\|$ for all $y\in B$, a contradiction. This completes the proof.

Received by the editors January 26, 1965.

[1] This research was partially supported by ONR Nonr-551(517).

Finally we shall show a negative example.

A negative example: Let $\mathfrak{H}$ be a separable Hilbert space, $\mathbf{C}$ the C^*-algebra of all compact operators on $\mathfrak{H}$, M a non-type I factor on $\mathfrak{H}$, then clearly $M \cap \mathbf{C} = (0)$.

Consider a C^*-subalgebra $D = M + \mathbf{C}$ in the C^*-algebra $B(\mathfrak{H})$ of all bounded operators on $\mathfrak{H}$, then $\mathbf{C}$ is an ideal of D.

Let ϕ be a pure state of D such that it is also pure on M then we shall consider the *-representation $\{\pi\phi, \mathfrak{H}\phi\}$ on a Hilbert space $\mathfrak{H}\phi$ of D constructed via ϕ.

If $\pi\phi(\mathbf{C})1\phi \neq 0$, where 1ϕ is the image of the unit 1 in $\mathfrak{H}\phi$, the closure of $\pi\phi(\mathbf{C})1\phi = \mathfrak{H}\phi$, for $\{\pi\phi, \mathfrak{H}\phi\}$ is irreducible and so $\mathfrak{H}\phi$ is separable, because $\mathbf{C}$ is uniformly separable.

Moreover the *-representation $\{\pi\tilde{\phi}, \mathfrak{H}\tilde{\phi}\}$ on a Hilbert space $\mathfrak{H}\tilde{\phi}$ of M constructed via the restriction $\tilde{\phi}$ of ϕ on M is also irreducible and clearly $\dim \mathfrak{H}\tilde{\phi} = \dim \{\text{the closure of } \pi\phi(M)1\phi\}$; hence $\dim \mathfrak{H}\tilde{\phi} \leqq \aleph_0$, so that $\{\pi\tilde{\phi}, \mathfrak{H}\tilde{\phi}\}$ is an irreducible *-representation on the separable Hilbert space $\mathfrak{H}\tilde{\phi}$.

On the other hand, the separable *-representation of M is always normal (cf. [2], [3]); hence $\{\pi\tilde{\phi}, \mathfrak{H}\tilde{\phi}\}$ is normal, so that the von Neumann algebra $\{\pi\tilde{\phi}(a) \mid a \in M\}$ on $\mathfrak{H}\tilde{\phi}$ is not of type I, a contradiction.

Therefore $\pi\phi(\mathbf{C})1\phi = 0$ and so $\phi(\mathbf{C}) = 0$.

Now let f be a state of D such that it is pure on M, $\mathfrak{F}_f = \{g \mid f = g \text{ on } M; g \text{ states of } D\}$.

Then $\mathfrak{F}_f$ is a convex compact subset of the state space of D. Let ψ be an arbitrary extreme point in $\mathfrak{F}_f$, then it is also extreme in the state space of D; hence by the above consideration, $\psi(\mathbf{C}) = 0$ and so $f(\mathbf{C}) = 0$. This shows that the problem is negative.

References

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