

TENSOR PRODUCTS OF LOCALLY CONVEX ALGEBRAS

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1. Introduction. The terms locally convex algebra, locally m -convex algebra, and Q -algebra are defined in [5]. Various cross-norms on the tensor product of Banach spaces are defined in [6]. The inductive and projective tensor products of locally convex spaces are defined in [4].

Gelbaum [1], [2] and Tomiyama [8] have remarked that the tensor product of two Banach algebras completed with respect to the greatest cross-norm is again a Banach algebra. When a Banach algebra is obtained by completing the tensor product of two commutative Banach algebras with respect to a cross-norm, these authors have given sufficient conditions under which its space of nonzero multiplicative functionals with relative weak* topology can be characterized as the topological direct product of the corresponding spaces associated with the two factors. Gil de Lamadrid [3] has announced the extension of this result to the noncommutative case. In [7], various necessary and sufficient conditions are given for the characterization of Gelbaum and Tomiyama to hold.

Lemmas 1 and 2 of this paper show that if a locally convex algebra is the completion, in a topology not stronger than the inductive topology, of the tensor product of two locally convex algebras, then there is a canonical continuous one-to-one mapping of its space of nonzero continuous multiplicative linear functionals, with the weak topology induced by the algebra, into the topological direct product of the corresponding spaces associated with the two factors. If this mapping is open, it provides a characterization of the space of nonzero continuous multiplicative functionals associated with the completed tensor product algebra. Theorem 1 shows that if the completed tensor product is a locally m -convex Q -algebra, then the mapping is open. Since every Banach algebra is a locally m -convex Q -algebra, this theorem generalizes the discussion of such characterizations given in [1], [3], [7], [8].

Theorem 2 shows the complete projective tensor product of locally m -convex algebras to be locally m -convex, while Theorem 3 states that if the two algebras are also complete commutative Q -algebras then the complete projective tensor product inherits these properties. Since the complete projective tensor product of locally convex spaces corresponds to the completion of the tensor product of Banach spaces

Received by the editors January 29, 1965.

with the greatest cross-norm, Theorems 2 and 3 are analogous to the result that such a product of Banach algebras is again one.

Theorem 4 gives a necessary and sufficient condition for semisimplicity of the complete projective tensor product of semisimple complete commutative locally m -convex Q -algebras. It generalizes a result for Banach algebras given in [7].

It is of interest to note that the arguments used to obtain Theorems 1 and 3 are quite different from those employed to obtain the corresponding results for Banach algebras. In particular the assumption of commutativity plays an essential role in the proof of Theorem 3, and the problem of extending the theorem to the noncommutative case remains open, while the analogous result for Banach algebras does not depend on commutativity. Theorem 3 provides a method for constructing new complete commutative locally m -convex Q -algebras from old ones. An example is given at the end of the paper.

2. Notation and remarks. We follow the notation and terminology of [5] for spectrum, $\Sigma_A(x)$, and spectral radius, $\sigma_A(x)$, of an element x in the algebra A . If A is a locally convex space, A' is the dual with the weak A topology. If A is also an algebra, $M(A)$ is the set of non-zero multiplicative functionals in A' with the relative topology.

REMARK 1. A locally convex algebra A is a Q -algebra iff $\sigma_A(x)$ is continuous at zero.

PROOF. Elementary from Proposition 13.5 of [5].

REMARK 2. If A is a complete locally m -convex algebra, then

$$\sigma_A(x) \geq \sup \{ |f(x)| : f \in M(A) \}$$

PROOF. If A is commutative the equality holds (Corollary 5.6 of [5]). Every commutative subalgebra of a complete locally m -convex algebra A is contained in a maximal commutative subalgebra B , which is closed and hence complete. If $x \in B$, then obviously $\Sigma_A(x) = \Sigma_B(x)$ and hence $\sigma_A(x) = \sigma_B(x)$. But $M(A) \subset M(B)$ in the sense that the restriction of an element of $M(A)$ is in $M(B)$; hence the inequality.

REMARK 3. A complete commutative locally m -convex algebra, A , is semisimple iff A' is the closed linear hull of $M(A)$.

PROOF. A is semisimple iff $m(a) = 0$ for all $m \in M(A)$ implies $a = 0$ (Corollary 5.5 of [5]). Suppose A' is the closed linear hull of $M(A)$. For any nonzero $a \in A$ there is f in A' such that $f(a) \neq 0$. For all $\epsilon > 0$, there is a linear combination g of elements of $M(A)$ such that $|g(a) - f(a)| < \epsilon$ so not all members of $M(A)$ can vanish at a . Hence A is semi-simple. Conversely if A is semisimple the closed linear hull of $M(A)$ is the orthogonal complement in A' of some closed subspace

of A . The closed subspace of A is zero by the cited corollary, so its orthogonal complement is A' .

3. Tensor products of locally convex algebras.

LEMMA 1. *Let A_3 be a locally convex algebra which is the completion of the tensor product, $A_1 \otimes A_2$, of two locally convex algebras in a topology not stronger than the inductive topology. Then each $m_3 \in M(A_3)$ is a continuous extension of $m_1 \otimes m_2$ where $m_1 \in M(A_1)$ and $m_2 \in M(A_2)$. Conversely, every such $m_1 \otimes m_2$ which has a continuous extension to A_3 is in $M(A_3)$.*

PROOF. The last assertion is immediate, for $m_1 \otimes m_2$ is multiplicative on $A_1 \otimes A_2$ and since multiplication is (separately) continuous in the completion, A_3 , a continuous extension must also be multiplicative. Suppose m_3 is a continuous nonzero multiplicative functional on A_3 . Since m_3 cannot vanish identically on $A_1 \otimes A_2$, there is $a_1 \otimes a_2$ with $m_3(a_1 \otimes a_2)$ not zero, and $m_3(a_1^3 \otimes a_2^3) = [m_3(a_1 \otimes a_2)]^3 \neq 0$. Letting m_i be defined by

$$\begin{aligned} m_1(b_1) &= m_3(b_1 a_1^3 \otimes a_2^3) / m_3(a_1^3 \otimes a_2^3), \\ m_2(b_2) &= m_3(a_1^3 \otimes b_2 a_2^3) / m_3(a_1^3 \otimes a_2^3) \end{aligned}$$

where $a_i, b_i, c_i \in A_i, i = 1, 2$; we have

$$\begin{aligned} m_1(b_1 c_1) &= m_3(b_1 c_1 a_1^3 \otimes a_2^3) / m_3(a_1^3 \otimes a_2^3) \\ &= \frac{m_3(b_1 \otimes a_2) m_3(a_1^3 \otimes a_2) m_3(c_1 \otimes a_2) m_3(a_1^3 \otimes a_2^3)}{m_3(a_1^3 \otimes a_2^3) m_3(a_1^3 \otimes a_2^3)} \\ &= \frac{m_3(b_1 a_1^3 \otimes a_2^3) m_3(c_1 a_1^3 \otimes a_2^3)}{m_3(a_1^3 \otimes a_2^3) m_3(a_1^3 \otimes a_2^3)} \\ &= m_1(b_1) m_1(c_1). \end{aligned}$$

By a similar calculation m_2 is multiplicative. Suppose $m_3(d_1 \otimes d_2) \neq 0$. Then an easy computation, using similar manipulations, yields

$$m_3(b_1 a_1^3 \otimes a_2^3) / m_3(a_1^3 \otimes a_2^3) = m_3(b_1 d_1^3 \otimes d_2^3) / m_3(d_1^3 \otimes d_2^3)$$

so m_1 is well defined. A corresponding identity shows m_2 to be well defined also. Now, by definition, $m_1 \otimes m_2(b_1 \otimes b_2) = m_1(b_1) m_2(b_2)$. By the preceding remark we can choose a_1 in the definition of m_1 and m_2 to be b_1 unless $m_3(b_1 \otimes a_2) = 0$ for all $a_2 \in A_2$. But in that case,

$$m_1(b_1) = m_3(b_1 \otimes a_2)m_3(a_1^3 \otimes a_2^2)/m_3(a_1^3 \otimes a_2^3) = 0$$

so

$$m_3(b_1 \otimes b_2) = 0 = m_1(b_1)m_2(b_2) = m_1 \otimes m_2(b_1 \otimes b_2).$$

Otherwise, choosing $a_1 = b_1$, we get

$$m_1 \otimes m_2(b_1 \otimes b_2) = \frac{m_3(b_1^4 \otimes a_2^3)}{m_3(b_1^3 \otimes a_2^3)} \frac{m_3(b_1^3 \otimes b_2 a_2^3)}{m_3(b_1^3 \otimes a_2^3)} = m_3(b_1 \otimes b_2).$$

By linearity and continuity of m_3 we have m_3 a continuous extension of $m_1 \otimes m_2$. By hypothesis, the topology of A_3 is not stronger than the inductive topology on $A_1 \otimes A_2$. By definition of the inductive topology, $m_3(b_1 \otimes b_2)$ is separately continuous in b_1 and b_2 , which is to say that the m_1 and m_2 defined above are continuous on A_1 and A_2 .

LEMMA 2. *Under the assumptions of Lemma 1 the mappings from $M(A_3)$ to $M(A_1)$ and $M(A_2)$ defined in the proof of Lemma 1, are continuous.*

PROOF. We prove only that the mapping from $M(A_3)$ to $M(A_1)$ is continuous. In the proof of Lemma 1, m_1 was defined as a function of m_3 by an equation equivalent to

$$m_1(b_1) = m_3(b_1 \otimes a_2)m_3(a_1^3 \otimes a_2^2)/m_3(a_1^3 \otimes a_2^3).$$

This can be written dually as

$$b_1(m_1) = [b_1 \otimes a_2(m_3)][a_1^3 \otimes a_2^2(m_3)]/[a_1^3 \otimes a_2^3(m_3)].$$

By definition of the weak topology, elementary theorems on continuity of combinations of complex valued functions and the fact that for each m_3 we can choose $a_1 \otimes a_2$ so that, on a neighborhood of m_3 , $m_3(a_1 \otimes a_2)$ is bounded from zero, we have the composite function $b_1(m_1(m_3))$ continuous in m_3 for each b_1 in A_1 . The topology of $M(A_1)$ is precisely the weak topology induced by the family of functions $b_1(m_1)$. The inverse images of open sets of complex numbers under these functions form a subbase for this topology. By continuity of the composite function, the inverse images of this subbase in $M(A_3)$ must be open. Thus the mapping from $M(A_3)$ to $M(A_1)$ is continuous.

THEOREM 1. *Let A_3 be a locally m -convex Q -algebra which is the completion, in a topology not stronger than the inductive topology, of the tensor product $A_1 \otimes A_2$ of two locally convex algebras. Then $M(A_3)$ is canonically homeomorphic to the subset of $M(A_1) \times M(A_2)$ consisting of elements (m_1, m_2) such that $m_1 \otimes m_2$ has a continuous extension to A_3 .*

PROOF. Lemmas 1 and 2 give the existence of a canonical one-to-one continuous map of $M(A_3)$ onto the designated subset of $M(A_1) \times M(A_2)$ with the relative product topology, so we need only show the continuity of the inverse. This inverse mapping carries (m_1, m_2) to m_3 , the continuous extension of $m_1 \otimes m_2$. Every neighborhood of m_3 in $M(A_3)$ contains an open set of the form

$$V = \{m'_3 : |m'_3(x_j) - m_3(x_j)| < \epsilon, \quad j = 1, 2, \dots, k\}$$

where x_j are members of A_3 . Since A_3 is a Q -algebra, by Remark 1 we have $\sigma_{A_3}(x)$ continuous at zero, so the set

$$U_j = \{y : \sigma_{A_3}(x_j - y) < \epsilon/4\}$$

is a neighborhood of x_j . Since $A_1 \otimes A_2$ is dense in A_3 , there is $y_j \in A_1 \otimes A_2$ in U_j such that $m_3(y_j) \neq 0$. By Remark 2 and the fact that A_3 is complete and locally m -convex, we have

$$(*) \quad |m'_3(x_j) - m'_3(y_j)| = |m'_3(x_j - y_j)| \leq \sigma_{A_3}(x_j - y_j) < \epsilon/4$$

for all m'_3 in $M(A_3)$.

Let

$$y_j = \sum_{i=1}^{n_j} a_{ij} \otimes b_{ij}.$$

Then

$$\begin{aligned} & |m'_3(y_j) - m_3(y_j)| \\ &= \left| \sum_{i=1}^{n_j} m'_1(a_{ij})(m'_2(b_{ij}) - m_2(b_{ij})) + (m'_1(a_{ij}) - m_1(a_{ij}))m_2(b_{ij}) \right|. \end{aligned}$$

Let $M = \max_{i,j} \{|m_1(a_{ij})|, |m_2(b_{ij})|\}$, $n = \max_j \{n_j\}$. Then if

$$|m'_1(a_{ij}) - m_1(a_{ij})| < \min \left\{ \frac{\epsilon}{4nM}, M \right\}$$

and

$$|m'_2(b_{ij}) - m_2(b_{ij})| < \frac{\epsilon}{8nM},$$

we have

$$|m'_3(y_j) - m_3(y_j)| < \frac{\epsilon}{2}.$$

Using the inequality $(*)$ satisfied in choosing y_j , we conclude that $|m'_3(x_j) - m_3(x_j)| < \epsilon$. Now let (m'_1, m'_2) be in the neighborhood of

(m_1, m_2) defined in the prescribed subset of $M(A_1) \times M(A_2)$ by

$$|m'_1(a_{ij}) - m_1(a_{ij})| < \min \left\{ \frac{\epsilon}{4nM}, M \right\}$$

and

$$|m'_2(b_{ij}) - m_2(b_{ij})| < \frac{\epsilon}{8nM}$$

for $i=1, 2, \dots, n_j; j=1, 2, \dots, k$. By assumption, $m'_1 \otimes m'_2$ has a continuous extension m'_3 and $m_1 \otimes m_2$ has the continuous extension m_3 . The preceding calculation shows that m'_3 lies in the open set V . The continuity of the inverse mapping has thus been shown.

4. Tensor products of locally m -convex algebras.

THEOREM 2. *The (complete) projective tensor product of two locally m -convex algebras is again locally m -convex.*

PROOF. An algebra is locally m -convex iff its topology is defined by a family of seminorms which are submultiplicative, i.e. which satisfy the multiplicative inequality for Banach algebra norms. Choosing such a family in each factor, repetition of the argument of Lemma 2 in [9] shows that the corresponding family of seminorms generating the topology of the projective tensor product (constructed in Proposition 2 of [4]) will also be submultiplicative. The completion is clearly also locally m -convex.

COROLLARY 1. *If A_3 is the complete projective tensor product of two locally m -convex algebras A_1 and A_2 and if A_3 is also a Q -algebra, then $M(A_3)$ is canonically homeomorphic to $M(A_1) \times M(A_2)$.*

PROOF. Theorems 1 and 2 and §1, no. 2 of [4].

THEOREM 3. *If A_1 and A_2 are complete commutative locally m -convex Q -algebras, then their complete projective tensor product is also such an algebra.*

PROOF. Let A_3 be the complete projective tensor product. In view of Theorem 2 and Remark 1, it suffices to show that σ_{A_3} is continuous at zero. By Lemma 1 and Corollary 5.6 of [5], we have

$$\begin{aligned} \sigma_{A_3}(x) &= \sup_{m_1, m_2} |m_1 \otimes m_2(x)| \\ &= \sup_{m_2} \sup_{m_1} |m_1(1 \otimes m_2(x))| \\ &= \sup_{m_2} \sigma_{A_1}(1 \otimes m_2(x)) \end{aligned}$$

where 1 is the identity map on A_1 and we have written the tensor products of operators to denote its extension, which always exists by §1, no. 2 of [4]. Since A_1 is a Q -algebra, σ_{A_1} is continuous at zero. There is a seminorm μ_1 , in the family defining the topology of A_1 such that $\sigma_{A_1} < \mu_1$. Thus

$$\sigma_{A_3}(x) \leq \sup_{m_2} \mu_1(1 \otimes m_2(x)).$$

Consider x in $A_1 \otimes A_2$, then

$$x = \sum_{i=1}^k x_{1i} \otimes x_{2i}$$

and

$$\begin{aligned} \sup_{m_2} \mu_1(1 \otimes m_2(x)) &\leq \sup_{m_2} \sum_{i=1}^k \mu_1(x_{1i}) |m_2(x_{2i})| \\ &\leq \sum_{i=1}^k \mu_1(x_{1i}) \sup_{m_2} |m_2(x_{2i})| \\ &\leq \sum_{i=1}^k \mu_1(x_{1i}) \sigma_{A_2}(x_{2i}). \end{aligned}$$

Again A_2 is a Q -algebra and so there is μ_2 such that $\sigma_{A_2} < \mu_2$ and

$$\sup_{m_2} \mu_1(1 \otimes m_2(x)) \leq \sum_{i=1}^k \mu_1(x_{1i}) \mu_2(x_{2i}).$$

Varying the representation of x and taking the infimum of the right side of this inequality, we obtain by Proposition 2 of [4] a seminorm, γ of the family defining the projective topology on $A_1 \otimes A_2$. For x in the dense subset $A_1 \otimes A_2$ we have

$$\mu_1(1 \otimes m_2(x)) \leq \gamma(x).$$

Since μ_1 is continuous on A_1 while $1 \otimes m_2$ and γ are continuous on A_3 , this inequality holds for all x . Thus we have

$$\sigma_{A_3}(x) \leq \gamma(x)$$

for γ a seminorm in the family defining the topology of A_3 , and σ_{A_3} is continuous at zero.

COROLLARY 2. *If A_1 and A_2 are complete commutative locally m -convex Q -algebras and A_3 is their complete projective tensor product, then $M(A_3)$ is canonically homeomorphic to $M(A_1) \times M(A_2)$.*

THEOREM 4. *Let A_1 and A_2 be semisimple complete commutative*

locally m -convex Q -algebras, and A_3 their complete projective tensor product. A_3 is semisimple iff the extension to A_3 of $A'_1 \otimes A'_2$ is dense in A'_3 .

PROOF. By Remark 3, A_3 is semisimple iff the closed linear hull of $M(A_3)$ is A'_3 . Let f be an element of A'_3 . If A_3 is semisimple there is, for every $\epsilon > 0$ and every finite set $\{x_i\}$ of elements of A_3 , a finite combination of elements of $M(A_3)$ approximating f within ϵ on the set $\{x_i\}$. By Corollary 2, each element of $M(A_3)$ is in the extension to A_3 of $A'_1 \otimes A'_2$ and this set is a linear space. Thus each neighborhood of f in A'_3 contains a member of $A'_1 \otimes A'_2$.

Conversely, suppose A'_3 is the closure of $A'_1 \otimes A'_2$. For any $x \neq 0$ in A_3 , there is f in A'_3 such that $f(x) \neq 0$. By hypothesis then, there are f_1 in A'_1 and f_2 in A'_2 such that $f_1 \otimes f_2(x) = f_1(1 \otimes f_2(x)) = f_2(f_1 \otimes 1(x)) \neq 0$. Since A_1 is semisimple, there is m_1 in $M(A_1)$ such that $m_1(1 \otimes f_2(x)) = m_1 \otimes f_2(x) = f_2(m_1 \otimes 1(x)) \neq 0$. Since A_2 is semisimple, there is then m_2 in $M(A_2)$ such that $m_2(m_1 \otimes 1(x)) = m_1 \otimes m_2(x) \neq 0$. Thus A_3 is semisimple.

As an application of these theorems, one might consider $L^1_D(G)$ the completion of the algebra of functions from a locally compact Abelian group, G , to the indefinitely differentiable functions of compact support, D , with the usual topology, which are absolutely summable. (See [4, p. 58] and [5, p. 13].) Grothendieck (in [4, §2, no. 2]) has shown that as a space $L^1_D(G)$ is topologically isomorphic with the complete projective tensor product of D and $L^1(G)$, the usual group algebra. (The extension to algebra isomorphism is easily carried out, as in [1].) From Theorem 3, we can conclude that $L^1_D(G)$ is a locally m -convex Q -algebra. Further the multiplicative functionals are homeomorphic to the direct product of the dual group of G with the real line by Corollary 2.

REFERENCES

1. B. Gelbaum, *Tensor products of Banach algebras*, Canad. J. Math. 11 (1959), 297-310.
2. ———, *Tensor products and related questions*, Trans. Amer. Math. Soc. 103 (1962), 525-548.
3. J. Gil de Lamadrid, *Uniform cross norms and tensor products of Banach algebras*, Bull. Amer. Math. Soc. 69 (1963), 797-803.
4. A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. No. 16 (1955), 331 pp.
5. E. A. Michael, *Locally multiplicatively-convex topological algebras*, Mem. Amer. Math. Soc. No. 11 (1952), 77 pp.
6. R. Schatten, *A theory of cross-spaces*, Annals of Mathematics Studies No. 26, Princeton Univ. Press, Princeton, N. J., 1950.

7. H. A. Smith, *Tensor products of topological algebras*, Dissertation, University of Pennsylvania, Philadelphia, Pa., 1964.

8. J. Tomiyama, *Tensor products of commutative Banach algebras*, Tôhoku Math. J. 12 (1960), 147–154.

9. T. Turumaru, *On the direct-product of operator algebra*. I, Tôhoku Math. J. 4 (1953), 242–251.

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SQUARE ROOTS IN BANACH ALGEBRAS

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A complex number which is not a nonpositive real number has a unique square root in the right half-plane. In this paper, we obtain an extension of this observation to general (complex) Banach algebras.

Since the elements we study are regular, and have logarithms, the existence of square roots is not at stake. Even the existence of roots having the desired spectral properties is evident. The uniqueness result we suppose to be new. It subsumes a somewhat weakened version of the classical result for positive, hermitian, bounded operators on a Hilbert space (that such an operator has a unique positive hermitian and bounded square root): our theorem would apply only to the positive definite (regular) case.

Whether the results of this paper can be extended to the case of not necessarily regular elements of a Banach algebra, we do not at present know.

DEFINITION. A subset A of the complex field will be called *positive* if $x \in A$ implies $x > 0$; A is *weakly positive* if $x \in A$ implies $\Re(x) > 0$. Let Σ denote the complement, in the complex plane, of the nonpositive real numbers.

THEOREM. Let \mathfrak{B} be a Banach algebra, and b an element of \mathfrak{B} with spectrum $\text{Sp}(b)$ contained in Σ . Then there exists in \mathfrak{B} a unique square root of b with weakly positive spectrum. If $\text{Sp}(b)$ is real (hence positive), so is $\text{Sp}(b^{1/2})$.

PROOF. EXISTENCE: We assume first that \mathfrak{B} has an identity ele-

Received by the editors May 24, 1965.