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A NOTE ON THE RECURSIVE UNSOLVABILITY OF PRIMITIVE RECURSIVE ARITHMETIC

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We wish to show the recursive unsolvability of primitive recursive arithmetic (PRA). By PRA we mean a quantifier-free formal system of arithmetic which has expressions for all primitive recursive functions. In such a system all valid variable free formulas are provable and both of the Gödel incompleteness theorems hold. Further, we may define in the system bounded quantifiers and (for a suitable Gödel numbering) the following primitive recursive functions: $\mathrm{th}(x)$, a function which enumerates the Gödel numbers of theorems of PRA, and $\mathrm{sub}(n, m)$, the function whose value is the Gödel number of the formula obtained by replacing the first variable in alphabetic order by the numeral n through the formula number m.[2]

If there is a recursive decision procedure for PRA, then the set of Gödel numbers of nontheorems is recursively enumerable. But if a set is recursively enumerable then it is primitive recursively enumerable. Thus if PRA is solvable there is a primitive recursive function whose range is precisely the set of Gödel numbers of nontheorems.

Assume there exists such a function f. Consider the formula

$$\mathrm{th}(x) = \mathrm{sub}(x_0, x_0) \supset (Ez).\ z \leqq x \,\&\, f(z) = \mathrm{sub}(x_0, x_0). \tag{1}$$

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[2] Detailed proofs may be found in J. R. Guard, *The independence of transfinite induction up to ω^ω in recursive arithmetic*, unpublished dissertation, Princeton University, 1962, or H. E. Rose, *On the consistency and undecidability of recursive arithmetic*, Z. Math. Logik Grundlagen Math. **7** (1961), 124–135.

Let the Gödel number of this formula be i, then the formula whose Gödel number is $\mathrm{sub}(i, i)$ will be

$$\mathrm{th}(x) = \mathrm{sub}(i, i) \supset (Ez).\ z \leqq x \,\&\, f(z) = \mathrm{sub}(i, i). \tag{2}$$

Suppose this formula is provable. There must be some k such that $\mathrm{th}(k) = \mathrm{sub}(i, i)$ is valid and hence provable. But if this formula and (2) are provable then by modus ponens and substitution $(Ez).\ z \leqq k \,\&\, f(z) = \mathrm{sub}(i, i)$. Thus $\mathrm{sub}(i, i)$ is one of the first k nontheorems, contrary to hypothesis.

Suppose (2) is not a theorem. By hypothesis, f enumerates all nontheorems, so there must be some n such that $f(n) = \mathrm{sub}(i, i)$. But now we may obtain a proof of (2) as follows: For each number m less than n, $\mathrm{th}(m) \neq \mathrm{sub}(i, i)$ will be provable, thus the conjunction of these n formulas will be provable. But this gives $\mathrm{th}(x) \neq \mathrm{sub}(i, i) \vee n \leqq x$, and hence $\mathrm{th}(x) \neq \mathrm{sub}(i, i) \vee n \leqq x \,\&\, f(n) = \mathrm{sub}(i, i)$, from which (2) follows immediately.

Thus the formula (2) is neither provable nor unprovable, so there must be no such formula and PRA is unsolvable.

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