NOTE ON AN IRRATIONAL POWER SERIES

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1. In several recent notes [1], [2], [3], [4] Mordell has investigated the series

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{a_n e(n\gamma)}{1 - xe(n\alpha)}, \tag{1}$$

where $|x| < 1$ and α, γ are real, and $e(\theta) = e^{2\pi i\theta}$. If $\sum |a_n|$ converges, the series (1) obviously converges and has the alternative expression

$$f(x) = \sum_{\nu=0}^{\infty} g(\nu\alpha + \gamma)x^\nu, \quad \text{where} \quad g(t) = \sum_{-\infty}^{\infty} a_n e(nt). \tag{2}$$

Further, if α is irrational and $x = re(k\alpha)$, where k is an integer, it is easily proved that

$$f(x) \sim \frac{a_{-k}e(-k\gamma)}{1-r} \quad \text{as } r \to 1 \quad (0 < r < 1). \tag{3}$$

The same proof shows that if $x = re(\theta)$, where θ is not congruent (mod 1) to an integral multiple of α, then $f(x) = o((1-r)^{-1})$.

Now suppose that in (1), and in any later sums over n, the terms n and $-n$ are taken together. Then Mordell has shown that there is another case, namely when

$$a_n = \begin{cases} 1/n & \text{if } n \neq 0, \\ 0 & \text{if } n = 0, \end{cases} \tag{4}$$

in which the series (1) converges and equals (2), and in which the limit relation (3) holds. In this particular case, as is well known,

$$g(t) = -2\pi i(t - [t] - \tfrac{1}{2}) \tag{5}$$

when t is not an integer. It is also well known that in this particular case the partial sums $\sum_{-N}^{N} a_n e(nt)$ of the series for $g(t)$ are uniformly bounded.

The object of the present note is to give a simple and direct proof of these results, in a slightly more general form:

THEOREM. *Suppose that $a_n \to 0$ as $n \to \pm\infty$, that $\sum_{-\infty}^{\infty} a_n$ converges, and that*

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$$(6) \qquad \sum_{-\infty}^{\infty} | a_n - a_{n+1} | \text{ converges.}$$

Suppose also that[1] *the partial sums* $\sum_{-N}^{N} a_n e(nt)$ *are uniformly bounded for all real t. Then for any real irrational* α *the two series for* $f(x)$ *in* (1) *and* (2) *converge and have the same sum, and the limit relation* (3) *holds.*

In §3 we prove analogous results for the double series

$$(7) \qquad \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} \frac{a_m a_n e(m\gamma + n\delta)}{1 - xe(m\alpha + n\beta)}$$

where 1, α, β are linearly independent over the rationals. This answers, at least partially, a question raised in [**3**].

2. **Proof of the theorem.** We observe first that the convergence of the series for $g(t)$ in (2) follows by partial summation from the first hypotheses of the theorem, for any nonintegral t. Further, if

$$(8) \qquad g_N(t) = \sum_{|n|>N} a_n e(nt),$$

partial summation gives

$$(9) \qquad | g_N(t) | < \delta(N) \| t \|^{-1}$$

where $\delta(N) \to 0$ as $N \to \infty$ and $\|t\|$ denotes the distance from t to the nearest integer.

The final hypothesis of the theorem now implies that $g(t)$ is bounded, so the first series in (2) converges for $|x| < 1$. Moreover we have

$$(10) \qquad | g_N(t) | < A$$

where A is independent of t and N (and similarly for $A_1, \cdots$, later).

The first series in (2) is

$$\sum_{\nu=0}^{\infty} x^\nu \left(\sum_{-N}^{N} a_n e(n\nu\alpha + n\gamma) + g_N(\nu\alpha + \gamma) \right)$$
$$= \sum_{-N}^{N} \frac{a_n e(n\gamma)}{1 - xe(n\alpha)} + \sum_{\nu=0}^{\infty} x^\nu g_N(\nu\alpha + \gamma).$$

Hence to establish the convergence of the series (1), and its equality

[1] This actually implies $a_n \to 0$, since it implies the convergence of $\sum |a_n|^2$.

with (2), it suffices to prove that

$$\sum_{\nu=0}^{\infty} |x|^{\nu} |g_N(\nu\alpha + \gamma)| \to 0 \quad \text{as} \quad N \to \infty$$

for any given x with $|x| < 1$. Here each single term $\to 0$ as $N \to \infty$, by (9) if $\nu\alpha + \gamma$ is not an integer, and by the convergence of $\sum a_n$ if $\nu\alpha + \gamma$ is an integer. Also, by (10),

$$\sum_{\nu=h}^{\infty} |x|^{\nu} |g_N(\nu\alpha + \gamma)| < A \frac{|x|^h}{1 - |x|},$$

and this is arbitrarily small if h is sufficiently large. Hence the result.

To prove the limit relation (3), we observe that the term $n = -k$ in (1) appears on the right of (3), and that any other single term is bounded as $r \to 1$, since α is irrational. Hence it suffices to prove that if $\epsilon > 0$ is given then

$$\left| \sum_{|n| > N} \frac{a_n e(n\gamma)}{1 - re(n\alpha + k\alpha)} \right| < \frac{\epsilon}{1 - r} \tag{11}$$

for any $N > N_0(\epsilon)$ and any r with $r_0(\epsilon) < r < 1$. The expression on the left is

$$\begin{aligned} &\leqq \sum_{\nu=0}^{\infty} r^{\nu} |g_N(\nu\alpha + \gamma)| \\ &< A_1 \sum_{\nu=0}^{\infty} r^{\nu} \min(1, \delta(N) \| \nu\alpha + \gamma \|^{-1}), \end{aligned} \tag{12}$$

by (9) and (10).

We choose a large integer q for which

$$\alpha = \frac{a}{q} + \frac{\theta}{q^2}, \qquad |\theta| < 1.$$

Write $\nu = qu + v$ where $u \geqq 0$ and $0 \leqq v < q$. Then

$$\begin{aligned} \| \nu\alpha + \gamma \| &> \left\| \frac{av}{q} + q\alpha u + \gamma \right\| - \frac{1}{q} \\ &> \left\| \frac{av + w(u)}{q} \right\| - \frac{3}{2q}, \end{aligned}$$

where $w(u)$ is the integer nearest to $q^2\alpha u + q\gamma$. For given u, we define v' (as a function of v) to be the absolutely least residue of $av + w(u)$

modulo q. In the sum (12) we take 1 in the minimum if $|v'| \leqq 2$, and find that the expression is

$$< A_1 \sum_{u=0}^{\infty} r^{qu} \left(5 + 2 \sum_{v'=3}^{q/2} \frac{\delta(N)q}{v' - 2} \right)$$

$$< A_2(1 + \delta(N)q \log q) \frac{1}{1 - r^q} \cdot$$

Hence (11) will hold provided that

$$A_2(1 + \delta(N)q \log q) < \epsilon(1 + r + r^2 + \cdots + r^{q-1}).$$

The right-hand side is greater than $\frac{1}{2}\epsilon q$ if $r^q > \frac{1}{2}$. We first choose q so that $A_2 < \frac{1}{4}\epsilon q$, then choose N_0 so that $A_2\delta(N) \log q < \frac{1}{4}\epsilon$ for $N > N_0$, then choose r_0 so that $r_0^q > \frac{1}{2}$. Then (11) holds, and this completes the proof of (3).

3. **The double series** (7). The proofs of the analogous results, under the hypotheses of the theorem, present no additional difficulty. We have

$$\sum_{\nu=0}^{\infty} x^\nu g(\nu\alpha + \gamma)g(\nu\beta + \delta)$$
$$= \sum_{m=-M}^{M} \sum_{n=-N}^{N} \frac{a_m a_n e(m\gamma + n\delta)}{1 - xe(m\alpha + n\beta)} + S_1 + S_2 - S_3,$$

where

$$S_1 = \sum_{\nu=0}^{\infty} x^\nu g(\nu\alpha + \gamma)g_N(\nu\beta + \delta)$$

and S_2 is a similar sum, and

$$S_3 = \sum_{\nu=0}^{\infty} x^\nu g_M(\nu\alpha + \gamma)g_N(\nu\beta + \delta).$$

Since $g(t)$ is bounded and $g_N(t)$ is uniformly bounded, and (as we have already proved)

$$\sum_{\nu=0}^{\infty} |x|^\nu |g_N(\nu\alpha + \gamma)| \to 0 \quad \text{as} \quad N \to \infty,$$

it follows that the double series (7) converges and has the sum $\sum x^\nu g(\nu\alpha+\gamma)g(\nu\beta+\delta)$.

As for the analogue of the limit relation, it suffices to prove that

$$\left| \sum_{\max(|m|,|n|)>N} \sum \frac{a_m a_n e(m\gamma + n\delta)}{1 - re(m\alpha + n\beta + k\alpha + l\beta)} \right| < \frac{\epsilon}{1-r}$$

for $N > N_0(\epsilon)$ and $r_0(\epsilon) < r < 1$. The double sum here can be expressed again as $S_1 + S_2 - S_3$, where x is now replaced by $re(k\alpha + l\beta)$, and now $M = N$. Since $g(t)$ is bounded and $g_N(t)$ is uniformly bounded, the desired estimate follows from what was proved in §2.

We can obviously replace a_n by b_n in (7), provided the sequence b_n satisfies the same conditions as a_n.

References

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