CYCLIC INJECTIVE MODULES OF FULL LINEAR RINGS[1]

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1. **Introduction.** Let L be the full ring of linear transformations on a vector space V_F of dimension $\aleph$ over a field F. If $\aleph$ is finite, then L is semi-simple Artin, and every L-module is both injective and projective. (See [**2**, Theorem 4.2, p. 11].) If $\aleph$ is infinite, then L is regular and right self injective [**5**, Theorem 5], and therefore every finitely generated right ideal is both injective and projective. However, it is shown in [**6**] that L has noninjective cyclic modules and nonprojective ideals in this case.

In this paper $\aleph$ will be assumed infinite. Some families of noninjective cyclic L-modules are found. In §2, the generalized continuum hypothesis is used to prove that if the cardinality of $F \leqq 2^{\aleph}$, then L has no nonzero cyclic injective module which is annihilated by the maximal ideal of L. This enables us to show that certain cyclic injective modules are isomorphic to right ideals of L. In §3, the techniques of [**6**] are used to prove that $(L/I)_L$ is not injective for any proper ideal I of L.

We adopt the following notational conventions. For each set A, $|A|$ denotes the cardinality of A. Every module M_L is a unital right L-module, and every map is an L-homomorphism. For each cardinal number $\aleph'$ such that $\aleph_0 \leqq \aleph' \leqq \aleph$, $J_{\aleph'}$ denotes the ideal of all elements of L of rank $< \aleph'$. Every proper ideal of L is known to be of the form $J_{\aleph'}$ for some such $\aleph'$. ([**4**, p. 93].) Upper case script letters will denote subsets of an indexing set for a basis of V_F. Indexing elements will be denoted by lower case Greek letters.

2. **Certain cyclic injective modules.** In this section we prove that certain cyclic injective modules must be isomorphic to direct summands of L_L; that is, they must be isomorphic to principal right ideals of L.

We observe the following:

(I) $|L| = |F|^{\aleph}$.

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(II) If $r \in J_{\aleph'}$, $L \supseteq H_L \supseteq J_{\aleph'}$, then $\mathrm{Hom}_L(rL, L/H) = 0$.

(III) If $L/H \neq 0$ and $r \in L$ is of rank $\aleph$, then $\mathrm{Hom}_L(rL, L/H) \neq 0$.

(IV) If $e = e^2 \in L$ and $x \in eL$, then there is a $y \in Le$ such that $xL = xyL$.

PROOF. (I) follows since L is naturally isomorphic to the ring of all column finite $\aleph \times \aleph$ matrices with entries in F. For (IV), if $xsx = x$, set $y = se$.

Since L is regular, $Lr = Le$ for some $e = e^2 \in L$, where rank e = rank r. In (II), let $\phi \in \mathrm{Hom}_L(rL, L/H)$. Then $\phi(r) = \phi(r)e = 0$, so $\phi = 0$. In (III), let $t \in L$ take eV onto V. Then te has right inverse, so $te \notin H$. We may then map $r \to te + H$.

If H is a right ideal of L, we say that a set of elements $\{r_i \in L\}$ is independent modulo H if the sum $\sum (r_i L + H)$ is direct in L/H.

Let $\{v_\iota \mid \iota \in \mathscr{I}\}$ be a basis of V. For any $\mathfrak{A} \subseteq \mathscr{I}$, let $e_{\mathfrak{A}}$ be the linear transformation defined by

$$e_{\mathfrak{A}}(v_\iota) = \begin{cases} 0 & \text{for } \iota \in \mathscr{I} - \mathfrak{A}, \\ v_\iota & \text{for } \iota \in \mathfrak{A}. \end{cases}$$

LEMMA 1. *A set of idempotents*

$$\{e_{\mathfrak{A}(\mu)} \mid \mathfrak{A}(\mu) \subseteq \mathscr{I},\ \mu \in \mathfrak{J}\}$$

is independent modulo $J_{\aleph'}$ if and only if for all $\mu, \nu \in \mathfrak{J}$,

$$|\mathfrak{A}(\mu)| \geqq \aleph', \textit{ and if } \mu \neq \nu,\ |\mathfrak{A}(\mu) \cap \mathfrak{A}(\nu)| < \aleph'.$$

PROOF. If $\{e_{\mathfrak{A}(\mu)} \mid \mathfrak{A}(\mu) \subseteq \mathscr{I},\ \mu \in \mathfrak{J}\}$ is independent modulo $J_{\aleph'}$, rank $e_{\mathfrak{A}(\mu)} = |\mathfrak{A}(\mu)| \geqq \aleph'$, and since

$$e_{\mathfrak{A}(\mu)} e_{\mathfrak{A}(\nu)} = e_{\mathfrak{A}(\nu)} e_{\mathfrak{A}(\mu)} = e_{\mathfrak{A}(\mu) \cap \mathfrak{A}(\nu)}$$

if $\mu \neq \nu$, $e_{\mathfrak{A}(\mu) \cap \mathfrak{A}(\nu)} \in J_{\aleph'}$, so $|\mathfrak{A}(\mu) \cap \mathfrak{A}(\nu)| < \aleph'$.

Conversely, if $e_{\mathfrak{A}(\mu)} r$ has rank $\geqq \aleph'$, and if $\mu \neq \nu$ implies $|\mathfrak{A}(\mu) \cap \mathfrak{A}(\nu)| < \aleph'$, then for $\{r_\sigma\} \subset L$, $\mu \notin \{\mu_\sigma\}$, rank $e_{\mathfrak{A}(\mu)} \sum_{\sigma=1}^{n} e_{\mathfrak{A}(\mu_0)}(r_\sigma) < \aleph'$. Thus

$$\text{rank } e_{\mathfrak{A}(\mu)} \left(e_{\mathfrak{A}(\mu)} r + \sum_{\sigma=1}^{n} e_{\mathfrak{A}(\mu_0)} r_\sigma \right) \geqq \aleph'$$

and $e_{\mathfrak{A}(\mu)} r + \sum_{\sigma=1}^{n} e_{\mathfrak{A}(\mu_0)} r_\sigma \notin J_{\aleph'}$. Thus $\{e_{\mathfrak{A}(\mu)} \mid \mathfrak{A}(\mu) \subseteq \mathscr{I},\ \mu \in \mathfrak{J}\}$ is independent modulo $J_{\aleph'}$.

Now assume the generalized continuum hypothesis; that is, for all ordinals α, $\aleph_{\alpha+1} = 2^{\aleph_\alpha}$. This assumption is not necessary if $\aleph = \aleph_0$. The following lemma is due to Tarski. (See [7, p. 201].)

LEMMA 2. *Every infinite set A can be decomposed into a class of sets, $\mathfrak{C}$, with $|\mathfrak{C}| = 2^{|A|}$, such that the sets of $\mathfrak{C}$ each have cardinality $|A|$ and pairwise intersections of cardinality $< |A|$.*

THEOREM 1. *If $|F| \leqq 2^{\aleph}$, then 0 is the only cyclic injective module annihilated by the maximal two-sided ideal of L.*

PROOF. By Lemma 2, we may divide a basis of V into a class $\mathfrak{C}$ consisting of sets of cardinality $\aleph$ with pairwise intersections of cardinality $<\aleph$, such that $|\mathfrak{C}| = 2^{\aleph}$. By Lemma 1, $\{e_{\mathfrak{A}(\mu)} \mid \mu \in \mathfrak{C}\}$ is independent modulo $J_{\aleph}$.

Let $H \supseteq J_{\aleph}$, $H \neq L$. By (III), each $e_{\mathfrak{A}(\mu)}$ for $\mu \in \mathfrak{C}$ may be mapped to zero or nontrivially into L/H, and these maps are 0 on $e_{\mathfrak{A}(\mu)} L \cap e_{\mathfrak{A}(\nu)} L \subseteq J_{\aleph}$ for $\mu \neq \nu$ by (II). Thus by Lemma 1, we may map each $e_{\mathfrak{A}(\mu)}$ independently of the others and extend to a well defined map from $\sum_{\mu \in \mathfrak{C}} e_{\mathfrak{A}(\mu)} L$ into L/H. This gives us at least $2^{2^{\aleph}}$ distinct maps. However, using (I),

$$|L/H| \leqq |L| = |F|^{\aleph} \leqq (2^{\aleph})^{\aleph} = 2^{\aleph},$$

so at most $2^{\aleph}$ of these maps are given by left multiplication by elements in L/H. Thus L/H is not injective ([**1**]).

We note that the proof of Theorem 1 also implies that $L/J_{\aleph}$ (as a ring) has no cyclic injective modules $\neq 0$. For the same class of idempotents gives rise to $2^{2^{\aleph}}$ $L/J_{\aleph}$-homomorphisms into any cyclic $L/J_{\aleph}$-module.

COROLLARY. *If $|F| \leqq 2^{\aleph_0}$ and V is of countable dimension, then any cyclic injective module is isomorphic to a right ideal of L.*

PROOF. If $H \subseteq L$, $L = H' \oplus M$, where H' is the injective hull of H in L (see [**3**]) and M is a right ideal of L. Then L/H is isomorphic to $M \oplus H'/H$, so if L/H is injective, so is H'/H. We note H'/H is isomorphic to $L/(H \oplus M)$. But L is an essential extension of $M \oplus H$, so $M \oplus H \supseteq J_{\aleph_0}$ = the socle of L. By Theorem 1, if $L/(M \oplus H)$ is injective, then $M \oplus H = L$, so L/H is isomorphic to M.

We note that this corollary does not depend on the generalized continuum hypothesis, since in the countable case it is not used in the proof of Lemma 2.

THEOREM 2. *Let L be the complete ring of linear transformations on an $\aleph$-dimensional vector space over a field of cardinality $\leqq 2^{\aleph_0}$. Then any simple injective module is isomorphic to a right ideal of L.*

PROOF. Let L/H be a nonzero simple injective L-module. Let e be an idempotent of smallest rank such that $e \notin H$.

Now $(eL+H)/H \approx eL/(eL\cap H)$. Let $N=eL\cap H$. Since $e\notin H$ and H is maximal, $eL+H=L$, so $L/H\approx eL/N$.

Assume eLe/Ne is not an injective eLe-module. Embed it in eL/N by $s+Ne\to s+N$ for all $s\in eLe$. This can be done since $Ne = eLe\cap N$.

Since eLe/Ne is not an injective eLe-module, there is a right ideal K of eLe and $\psi\in \mathrm{Hom}_{eLe}(K, eLe/Ne)$ such that ψ does not extend to an element of $\mathrm{Hom}_{eLe}(eLe, eLe/Ne)$. (See Baer [1]; or [2, p. 8].)

Set $M=KL$, and define ϕ: $M\to eL/N$ by

$$\phi\left(\sum_{\sigma=1}^{n} i_\sigma r_\sigma\right) = \sum_{\sigma=1}^{n} \psi(i_\sigma) r_\sigma$$

for $i_\sigma\in K$, $r_\sigma\in L$. ϕ is clearly an L-homomorphism if it is well-defined.

Let $\sum_{\sigma=1}^{n} i_\sigma r_\sigma=0$. Since L is regular, there is an $f=f^2\in L$ such that $fL=\sum_{\sigma=1}^{n} i_\sigma L\subseteq eL$. Then $fe=\sum_{\sigma=1}^{n} i_\sigma s_\sigma e=\sum_{\sigma=1}^{n} i_\sigma e s_\sigma e\in K$, and $fei_\sigma = i_\sigma$. Hence $\psi(i_\sigma)=\psi(fe)i_\sigma$. Then

$$\phi\left(\sum_{\sigma=1}^{n} i_\sigma r_\sigma\right) = \sum_{\sigma=1}^{n} \psi(fe) i_\sigma r_\sigma = \psi(fe)\sum_{\sigma=1}^{n} i_\sigma r_\sigma = 0.$$

Thus ϕ is well-defined.

Now eL/N is an injective L-module, so ϕ can be extended to an element $\chi\in\mathrm{Hom}_L(L, eL/N)$. Then χ restricted to eLe extends ψ to an element of $\mathrm{Hom}_{eLe}(eLe, eLe/Ne)$, a contradiction. We thus conclude that eLe/Ne is an injective eLe-module.

Let $s\in eLe$, rank $s<$rank e. Then $s\in H$ since rank e was the smallest rank of an element not in H. Hence $s\in eLe\cap H=Ne$. Since $eLe = \mathrm{Hom}_F(eV, eV)$ is the ring of all linear transformations on the vector space eV of dimension $=$rank e, and eLe/Ne is a nonzero injective eLe-module whose kernel contains all transformations of rank $<$rank e, by Theorem 1, rank e cannot be infinite. Then rank $e=1$ since eL is a direct sum of submodules of rank 1, and these cannot all be contained in H since $e\notin H$. Then $L/H\approx eL$, a simple injective right ideal of L.

We observe that if H is a right ideal of L, e an idempotent of smallest rank not in H, and $eH\subseteq H$, then $eL/(H\cap eL)$ is not injective by the proof of Theorem 2, and since $eL/(H\cap eL)=eL/eH$ is a direct summand of L/H, L/H is not injective.

3. L **modulo two-sided ideals.** If I is a proper two-sided ideal of L, we may use the machinery of [6] to prove that L/I is not an injective right L-module.

THEOREM 3. *If I is a proper two-sided ideal of L, then L/I is not an injective module.*

PROOF. $I=J_{\aleph'}$ for some $\aleph'$ with $\aleph_0\leqq\aleph'\leqq\aleph$. Divide a basis of V, $\{v_\iota \mid \iota\in\mathfrak{g}\}$, into $\aleph$ disjoint subsets $\{\mathfrak{A}(\mu)\mid\mu\in\mathfrak{I}\}$ such that $|\mathfrak{A}(\mu)|=\aleph$ for all $\mu\in\mathfrak{I}$. Then $\{e_{\mathfrak{A}(\mu)}\mid\mu\in\mathfrak{I}\}$ is independent in L/I. Since independence modulo I depends only on finite sets of idempotents, we may use Zorn's Lemma to enlarge this set to a maximal set $S=\{e_{\mathfrak{B}(\nu)}\mid\nu\in\mathfrak{K}\}$ of idempotents independent modulo I.

Let $K=\sum_{\nu\in\mathfrak{K}} e_{\mathfrak{B}(\nu)}L$. Define $\phi\colon K\to L/I$ by

$$\phi(e_{\mathfrak{A}(\mu)}) = e_{\mathfrak{A}(\mu)} + I \qquad \text{for all } \mu\in\mathfrak{I},$$
$$\phi(e_{\mathfrak{B}(\nu)}) = 0 \qquad \text{for all } \nu\in\mathfrak{K}-\mathfrak{I}.$$

Since the idempotents of S are independent modulo I, ϕ extends by linearity to an L-homomorphism from K into L/I.

Assume ϕ is induced by some $\psi\in\operatorname{Hom}_L(L, L/I)$. Let $\psi(1)=m+I$. Then

$$\text{(i)} \qquad me_{\mathfrak{A}(\mu)} - e_{\mathfrak{A}(\mu)} \in I \qquad \text{for all } \mu\in\mathfrak{I},$$
$$\text{(ii)} \qquad me_{\mathfrak{B}(\nu)} \in I \qquad \text{for all } \nu\in\mathfrak{K}-\mathfrak{I}.$$

If $r\in I$, $|\{e_{\{\iota\}})\mid e_{\{\iota\}})r\neq 0\}|\leqq\aleph_0\cdot\text{rank } r$ (or is finite) $<\aleph'$ since at most a finite number of $e_{\{\iota\}}$ fail to annihilate each vector in a basis of range r. If we pre-multiply and post-multiply the elements in (i) and (ii) by $e_{\{\iota\}}$, then from (i) we conclude that

$$\text{(i*)} \qquad |\{\iota\in\mathfrak{A}(\mu)\mid e_{\{\iota\}}me_{\{\iota\}}\neq e_{\{\iota\}}\}| < \aleph' \quad \text{for all } \mu\in\mathfrak{I}$$

and from (ii) we conclude that

$$\text{(ii*)} \qquad |\{\iota\in\mathfrak{B}(\nu)\mid e_{\{\iota\}}me_{\{\iota\}}\neq 0\}| < \aleph' \quad \text{for all } \nu\in\mathfrak{K}-\mathfrak{I}.$$

For each $\mu\in\mathfrak{I}$, select $\iota(\mu)\in\mathfrak{A}(\mu)$ such that

$$e_{\{\iota(\mu)\}}me_{\{\iota(\mu)\}} = e_{\{\iota(\mu)\}}.$$

This is possible since $|\mathfrak{A}(\mu)|=\aleph\geqq\aleph'$, and (i*) holds.

Since S is a maximal independent set, the idempotents in $S\cup\{e_{\{\iota(\mu)\}\mid\mu\in\mathfrak{I}}\}$ cannot be independent modulo I. Hence, by Lemma 1, $\{\iota(\mu)\mid\mu\in\mathfrak{I}\}$ must have at least $\aleph'$ elements in common with $\mathfrak{B}(\nu)$ for some $\nu\in\mathfrak{K}$. $\nu\notin\mathfrak{I}$ since $\iota(\mu)\notin\mathfrak{A}(\nu)$ for μ, $\nu\in\mathfrak{I}$, $\mu\neq\nu$. (The $\mathfrak{A}(\mu)$, $\mu\in\mathfrak{I}$ are pairwise disjoint.) Therefore (ii*) above applies to $\mathfrak{B}(\nu)$. But this implies that

$$\aleph' \leqq |\{\iota(\mu)\in\mathfrak{B}(\nu)\mid e_{\{\iota(\mu)\}}me_{\{\iota(\mu)\}} = e_{\{\iota(\mu)\}}\neq 0\}|,$$

contradicting (ii*). Thus ϕ does not extend to L, and L/I is not injective.

COROLLARY. *L/I is not a right self injective ring.*

PROOF. The homomorphism ϕ defined in the proof of Theorem 3 induces an L/I-homomorphism of K/I into L/I which is not extendable to L/I.

We observe that Theorem 3 can be slightly generalized. If H is a right ideal of L containing any transformations of rank $\aleph'$, then H contains a projection e onto a space of dimension $\aleph'$. Now select the idempotents in Theorem 3 from the ring eLe, replacing $\aleph$ by $\aleph'$. Then one constructs a homomorphism $\phi: eL \rightarrow H/(H\cap I)$ which is not induced by left multiplication by an element in L/I, a priori not in $H+I/I \approx H/(I\cap H)$, so $H/(I\cap H)$ is not injective. (Of course, $H/(I\cap H)$ need not be cyclic.)

4. **Concluding remarks.** Infinite full rings have several families of noninjective cyclic modules. In the countable dimensional case, the only cyclic injective modules of a full linear ring over a "small" field are principal right ideals. Is it possible to remove the cardinality restriction on the field? A noncounting argument proving that L modulo any right ideal containing the maximal two-sided ideal is not injective would enable one to eliminate both the cardinality restrictions and generalized continuum hypothesis used in §2. The author has been unable to develop such an argument.

In [6], the author showed that any right self injective regular ring modulo any right ideal generated by an infinite set of orthogonal idempotents was not an injective module. This result gives an additional family of noninjective cyclic L-modules. Moreover, if H is a right ideal of L, e an idempotent of smallest rank such that $e \notin H$, and if $eH = eL\cap H$, then the map lifting argument used in Theorem 3 may be used to prove that L/H is not injective, for we have eL/eH is a direct summand of L/H.

Since L is right self injective but not left self injective, there is no reason to suspect that L has cyclic injective left modules. We observe that all proofs of noninjectivity used here work equally well on the left. Conclusions about cyclic injective left L-modules are as for right modules or must be modified to read that over suitably small fields, L has no cyclic injective left modules in the countable dimensional case, and no simple injective left modules in the more general case.

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