ON THE ASYMPTOTIC GROWTH OF SOLUTIONS TO A NONLINEAR EQUATION

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We shall consider the nonlinear integral equation

$$\dot{x}(t) - \dot{x}(0) + \int_0^t a(s)g(x(s))\,ds = h(t), \qquad 0 \leqq t < \infty, \tag{1}$$

where $a(t)$ and $h(t)$ are continuous for $t \geqq 0$ and $g(x)$ is continuous for all x. If $h(t)$ is absolutely continuous on bounded sets, then (1) is equivalent to the differential equation

$$\ddot{x}(t) + a(t)g(x(t)) = \dot{h}(t), \tag{2}$$

so our results will hold for solutions of (2). We shall be primarily interested in the nonlinear oscillator, i.e. $a(t) > 0$, $xg(x) > 0$ for $x \neq 0$, but in Theorem III below, sign restrictions are removed.

If we consider $h(t)$ to be a sample function of a Brownian motion process $h(t, \omega)$ on a probability space Ω, we see the motivation for considering (1). Equation (2) then represents a nonlinear oscillation driven by "white noise," an illegitimate process in the sense that for almost all $\omega \in \Omega$, $h(t, \omega)$ is not differentiable.

For an account of the probabilistic aspects of (1) for the case in which $a(t)$ is constant, see [**2**]. We shall be concerned with bounds on the asymptotic growth of solutions to (1). Various results for the homogeneous case are found in [**4**], and [**5**]. Results for the nonhomogeneous case when $a(t)$ is constant are obtained for (1) in [**2**] and for (2) in [**3**].

Note that the usual local existence theorems which hold for (2) may be obtained for (1) by considering the equivalent system of equations

$$\begin{aligned} \dot{x}(t) &= u(t) + h(t), \\ -\dot{u}(t) &= a(t)g(x(t)) \end{aligned}$$

as in [**2**]. Here, of course, $u(t)$ is defined to be $\dot{x}(t) - h(t)$.

The proofs and results of Theorems I and II are essentially the same as those of Waltman in [**4**] for the homogeneous case, and depend on the well-known lemma due to Gronwall [**1**, p. 37].

THEOREM I. *Suppose $g(x)$ is a monotone nondecreasing odd function*

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which is positive for $x>0$ and continuous for all x. Suppose also that $a(t)$ is positive and absolutely continuous on bounded sets, and that

$$\int_0^\infty |a'(s)|/a(s)\,ds < \infty. \tag{3}$$

Let $x(t)$ be a solution to (1) defined at $t=0$. Then $x(t)$ can be extended to $[0, \infty)$, and

$$x(t) = O\left(t + \int_0^t |h(s)|\,ds\right)$$

as t goes to infinity.

PROOF. Condition (3) implies that $a(t)$ is bounded below by some positive number. In fact, the following is obvious:

LEMMA I. *With $a(t)$ as above, $\sup_{0\leq t\leq\tau<\infty}[a(t)/a(\tau)]$ is finite. Further, let $\{T_i\}$ be a nondecreasing sequence of positive numbers and let*

$$\alpha_i = \sup_{T_i\leqq t\leqq\tau\leqq T_{i+1}} [a(t)/a(\tau)].$$

Then $\prod_{i=1}^\infty \alpha_i<\infty$.

Let $\beta=\sup_{t\geq 0} a(t)^{-1}$. The theorem will follow easily from the next lemma.

LEMMA II. *Let $[A, B]$ be an interval on which $x(t)$ is defined and such that $\dot{x}(t)-h(t)$ does not change sign on this interval. Let*

$$\alpha_{A,B} = \sup_{A\leqq t\leqq\tau\leqq B}[a(t)/a(\tau)],$$

$$K_{A,B} = \alpha_{A,B}\exp\left\{\int_A^B |a'(s)|/a(s)\,ds\right\},$$

$$M_{A,B} = \int_A^B |h(s)|\,ds.$$

Pick $\mu\geqq 0$ so that

$$\int_0^\mu g(s)\,ds = (2\alpha_{A,B})^{-1}\beta(\dot{x}(A) - h(A))^2.$$

Then

$$|x(B)| \leqq K_{A,B}(|x(A)| + \mu) + (2 + K_{A,B})M_{A,B}.$$

PROOF OF LEMMA II. Assume for definiteness that $\dot{x}(t)-h(t)$ is nonnegative on $[A, B]$. Let

$$H(t) = \int_A^t h(s)\, ds \quad \text{and} \quad y(t) = x(t) - H(t).$$

Then

$$\ddot{y}(t) = -a(t)g(y(t) + H(t)) \leqq -a(t)g(y(t) - M_{A,B}).$$

Therefore, since $\dot{y}(t) \geqq 0$ on $[A, B]$,

$$\dot{y}(t)^2/2 \leqq \dot{y}(A)^2/2 - \int_A^t a(s)g(y(s) - M_{A,B})\dot{y}(s)\, ds \tag{4}$$

for $A \leqq t \leqq B$.

Set $G(x) = \int_0^x g(s)ds$ and note that $G(x) > 0$ for $x \neq 0$. Integrating by parts in (4), we have

$$\begin{aligned} a(t)G(y(t) - M_{A,B}) &\leqq \dot{y}(A)^2/2 + a(A)G(y(A) - M_{A,B}) \\ &\quad + \int_A^t \frac{|a'(s)|}{a(s)} a(s)G(y(s) - M_{A,B})\, ds \end{aligned}$$

for $A \leqq t \leqq B$. By Gronwall's inequality,

$$\begin{aligned} a(B)G(y(B) - M_{A,B}) &\leqq [\dot{y}(A)^2/2 + a(A)G(y(A) - M_{A,B}] \\ &\quad \cdot \exp\left\{\int_A^B |a'(s)|/a(s)\, ds\right\}. \end{aligned}$$

Recalling that $g(x)$ is an odd, nondecreasing function, we see that

$$\begin{aligned} \int_0^{|y(B)-M_{A,B}|} g(s)\, ds &\leqq K_{A,B}\left[\beta(\dot{y}(A)^2/2)\alpha_{A,B}^{-1} + \int_0^{|y(A)-M_{A,B}|} g(s)\, ds\right] \\ &\leqq K_{A,B}\int_0^{|y(A)|+M_{A,B}+\mu} g(s)\, ds \end{aligned} \tag{5}$$

where we use the fact that $\dot{y}(A) = \dot{x}(A) - h(A)$.

Now suppose p and q are positive numbers and λ is a number $\geqq 1$. Suppose also that

$$\int_0^p g(s)\, ds \geqq \frac{1}{\lambda}\int_0^q g(s)\, ds.$$

Since $g(x)$ is nondecreasing and positive for $x > 0$, it follows that $p \geqq q/\lambda$. From (5), then, we have

$$|y(B)| \leqq K_{A,B}(|y(A)| + \mu) + (1 + K_{A,B})M_{A,B}.$$

Since $|x(t)| = |y(t)+H(t)| \leqq |y(t)| + M_{A,B}$ and $x(A) = y(A)$, the conclusion of Lemma II is immediate.

Now suppose there is a $T \geqq 0$ such that $x(t)$ can be extended past T and such that in the largest interval $[T, N)$, $N \leqq \infty$, to which $x(t)$ can be extended, $\dot{x}(t) - h(t)$ does not change sign. If $N < \infty$, then Lemma II applied to the closed sub-intervals of $[T, N)$ shows that $x(t)$ is bounded for $t < N$. Then (1) implies that $\dot{x}(t)$ is also bounded for $t < N$. But this implies that $x(t)$ can be extended to $[0, N]$, contradicting the definition of N. Thus $N = \infty$. Theorem I follows in this case by applying Lemma II to finite intervals $[T, t]$ and letting t go to infinity.

If such a T does not exist, then there must be an unbounded set of zeros of $\dot{x}(t) - h(t)$. In this case we define two disjoint sets S^1 and S^2, with $[0, \infty) = S^1 \cup S^2$, as follows:

$$S^1 = \{t \geqq 0 \mid \text{There is a neighborhood } (a, b) \text{ of } t \text{ in which } \dot{x}(\cdot) - h(\cdot) \text{ has only a finite number of zeros}\}.$$

$$S^2 = \{t \geqq 0 \mid t \notin S^1\}.$$

If $t \in S^2$, then there is a sequence t_i approaching t with $\dot{x}(t_i) - h(t_i) = 0$. It follows that $d(\dot{x}(t) - h(t))/dt = 0$ and by (1), $x(t) = 0$.

For each $t \in S^1$, define I_t to be the intersection of $[0, \infty)$ with the union of all neighborhoods of t in which $\dot{x}(\cdot) - h(\cdot)$ has only a finite set of zeros. Supose first that for some t_1, I_{t_1}, is semi-infinite. Then there must exist an increasing and unbounded sequence $\{T_i\}$ with $(-1)^i(\dot{x}(t) - h(t)) \geqq 0$ on $[T_i, T_{i+1}]$, $i = 1, 2, \cdots$.

Let $\gamma_i = K_{T_i, T_{i+1}}$ and note that $\dot{x}(T_i) = h(T_i)$. Lemma II implies that for $i = 1, 2, \cdots$

$$|x(T_{i+1})| \leqq \gamma_i |x(T_i)| + (2 + \gamma_i) M_{T_i, T_{i+1}}.$$

Lemma I and condition 3 show that $L = \prod_{i=1}^{\infty} \gamma_i$ is finite. It is easy to show by induction that for $n = 2, 3, \cdots$,

$$\begin{aligned} |x(T_n)| &\leqq \left(\prod_{i=1}^{n-1} \gamma_i\right) |x(T_1)| + 3\left(\prod_{i=1}^{n-1} \gamma_i\right) \int_{T_1}^{T_n} |h(s)|\, ds \\ &\leqq L |x(T_1)| + 3L \int_{T_1}^{T_n} |h(s)|\, ds, \end{aligned}$$

and Theorem I follows easily in the same way as when $\dot{x}(t) - h(t)$ does not change sign for $t \geqq T$.

Suppose, on the other hand, that I_t is bounded for each $t \in S^1$. If, for some $t \in S^1$, $I_t = [0, b)$, $b < \infty$, then $[0, t]$ contains only a finite number of zeros of $\dot{x}(\cdot)-h(\cdot)$. The argument above shows that

$$|x(t)| \leqq L_1(|x(0)| + |\mu_{0,t}|) + 3L_1\int_0^t |h(s)|\, ds$$

where

$$\int_0^{\mu_{A,B}} g(s)\, ds = (2\alpha_{A,B})^{-1}\beta(\dot{x}(A) - h(A))^2$$

$$L_1 = \left[\sup_{0\leqq t\leqq \tau<\infty} a(t)/a(\tau)\right]\exp\left\{\int_0^\infty |a'(s)|/a(s)\, ds\right\}$$

and $\alpha_{A,B}$ and β are as on p. 41, for any $B\geqq A\geqq 0$.

Finally, suppose that for some t, $I_t=(a, b)$, $a\geqq 0$, $b<\infty$. Then $a\in S^2$ and $x(a)=0$. For any c in (a, t), $[c, t]$ contains only a finite number of zeros of $\dot{x}(\cdot)-h(\cdot)$. Therefore, as before,

$$|x(t)| \leqq L_1(|x(c)| + |\mu_{c,t}|) + 3L_1\int_c^t |h(s)|\, ds.$$

Letting c approach a, we see that

$$|x(t)| \leqq 3L_1\int_a^t |h(s)|\, ds.$$

We have shown that there is an M such that for any

$$t\geqq 0, \quad |x(t)| \leqq M + 3L_1\int_0^t |h(s)|\, ds,$$

and this completes the proof of Theorem I.

THEOREM II. *Suppose $a(t)$ is positive, absolutely continuous on bounded sets, and nondecreasing, and $g(x)$ satisfies the same conditions as in Theorem* I. *If $x(t)$ is a solution to* (1) *defined at $t=0$, then $x(t)$ can be extended to $[0, \infty)$ and there is a constant c such that*

$$|x(t)| \leqq c + 3\int_0^t |h(s)|\, ds, \quad \textit{all } t \geqq 0.$$

PROOF. Let $[A, B]$, $H(t)$, $M_{A,B}$ and $y(t)$ be as in Lemma II. Suppose $\dot{x}(t)-h(t)\geqq 0$ on $[A, B]$. Just as in Lemma II, if $A\leqq t\leqq B$,

$$a(t)G(y(t) - M_{A,B}) \leqq a(A)G(y(A) - M_{A,B}) + \dot{y}(A)^2/2 + \int_A^t \frac{a'(s)}{a(s)} a(s)G(y(s) - M_{A,B})\, ds.$$

In this case, Gronwall's inequality implies that

$$a(B)G(y(B) - M_{A,B}) \leqq [a(A)G(y(A) - M_{A,B}) + y(A)^2/2]a(B)/a(A).$$

From this,

$$|y(B)| \leqq |y(A)| + \mu + 2M_{A,B}$$

where

$$\int_0^\mu g(s)\,ds = (\dot{y}(A)^2/2)a(A)^{-1}, \qquad \mu \geqq 0.$$

(The same result holds if $\dot{x}(t) - h(t) \leqq 0$ on $[A, B]$.)

Since $|x(B)| \leqq |y(B)| + M_{A,B}$ and $x(A) = y(A)$,

$$|x(B)| \leqq |x(A)| + \mu + 3M_{A,B}.$$

The rest of the proof of Theorem II proceeds much as that of Theorem I and will be omitted.

We note that in both of these theorems, if $h(t)$ is constant, then all solutions are bounded. (See [**4**].) We conjecture that the boundedness of all solutions to $\ddot{x}+a(t)g(x)=0$ is sufficient to insure that any solution to (1) is $O(1+\int_0^t|h(s)|ds)$ as t goes to infinity.

If $a(t)$ is not bounded below by a positive number, then the conclusion of Theorem I does not hold. In fact, if $\int^\infty sa(s)ds < \infty$, then there is a solution to $\ddot{x}+a(t)x=0$ which is asymptotic to a straight line of positive slope [**1**, p. 103]. In this case, the following theorem provides a bound in the (very restrictive) case $|g(x)| \leqq k|x|^\alpha$ for some constant k and $\alpha \in [0, 1]$. We can remove the restrictions on the signs of $a(t)$ and $g(x)$ in this case.

We might comment here on a result in [**5**] to the effect that if $\int^\infty t^\alpha |a(t)| dt < \infty$ and $|g(x)| \leqq k|x|^\alpha$ for some k and $\alpha \geqq 0$, then all solutions to $\ddot{x}+a(t)g(x)=0$ have bounded derivatives. In fact, this is false unless we restrict α to $[0, 1]$. For example, with $a(t) = -2/t^4$, $g(x)=x^2$, the function $x(t)=t^2$ is such a solution.

THEOREM III. *Suppose* $|g(x)| \leqq k|x|^\alpha$ *for some* k *and some* $\alpha \in [0, 1]$. *Suppose further that* $a(t)$ *is continuous for* $t \geqq 0$ *and* (*) $\int_0^\infty t|a(t)|dt < \infty$. *Let* $x(t)$ *be any solution to* (1) *defined at* $t=0$. *Then* $x(t)$ *can be extended to* $[0, \infty)$, *and* $x(t) = O(t+t|\int_0^t h(s)ds|)$ *as* t *goes to infinity.*

PROOF. We can write, as far as $x(t)$ is defined,

$$x(t) = x(0) + t\dot{x}(0) + \int_0^t h(s)\,ds - \int_0^t (t-s)a(s)g(x(s))\,ds,$$

and if we let $H(t)=\int_0^t h(s)ds$, then there are numbers c_1 and c_2 such that

$$
\begin{aligned}
&|x(t)| + 1 \leqq c_1 + c_2 t + |H(t)| \\
(6) \qquad &\qquad + \int_0^t (t-s)k|a(s)|(|x(s)|+1)\,ds.
\end{aligned}
$$

Lemma III. *Suppose that $\phi(t)$ is an integrable function on $[0, T]$, and for t in this interval,*

$$\phi(t) \leqq A(t) + \int_0^t (t-s)B(s)\phi(s)\,ds,$$

where $B(t)$ is continuous and nonnegative and $A(t)$ is continuous. Let $z(t)$ be any solution to $\ddot{z}-B(t)z=0$ which is positive for all $t\geqq 0$. Then for

$$t\in[0,T],\quad \phi(t) \leqq A(t) + z(t)\int_0^t\int_0^s A(u)B(u)z(u)/z(s)^2\,du\,ds.$$

Proof of Lemma III. Let

$$R(t) = \int_0^t (t-s)B(s)\phi(s)\,ds.$$

Then $\ddot{R}(t)=B(t)\phi(t)\leqq B(t)A(t)+B(t)R(t)$. Let

$$Q(t) = \int_0^t B(s)z(s)\,ds + \dot{z}(0).$$

Then we have $d[z(t)\dot{R}(t)-Q(t)R(t)]/dt\leqq B(t)A(t)z(t)$. Integrate, divide by $z(t)$ and multiply by $\exp\{-\int_0^t Q(s)/z(s)ds\}$, and we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left(R(t)\exp\left\{-\int_0^t Q(s)/z(s)\,ds\right\}\right) \\
&\qquad \leqq \exp\left\{-\int_0^t Q(s)/z(s)\,ds\right\}\cdot\int_0^t B(s)A(s)z(s)/z(t)ds.
\end{aligned}
$$

The result follows by integrating again and noting that

$$Q(t) = \int_0^t B(s)z(s)\,ds + \dot{z}(0) = \int_0^t \ddot{z}(s)\,ds + \dot{z}(0) = \dot{z}(t).$$

Now we note that condition (*) insures the existence of a solution $z_1(t)$ to $\ddot{z}(t)-k|a(t)|z(t)=0$, with $\eta\leqq z_1(t)\leqq 1$ for some $\eta>0$, all $t\geqq 0$.

(See [1, p. 103].) Using Lemma III with (6) gives

$$
\begin{aligned}
|x(t)|+1 \leqq c_1+c_2t+|H(t)| & \\
&+z_1(t)\int_0^t \frac{1}{z_1(s)^2}\int_0^s k\,|a(u)|\,z_1(u)[c_1+c_2u+|H(u)|]\,du\,ds \\
&\leqq c_1+c_2t+|H(t)| \\
&\quad +(k/\eta^2)\int_0^t (t-s)\,|a(s)|\,[c_1+c_2s+|H(s)|]\,ds.
\end{aligned}
$$

Clearly $x(t)$ can be extended to $[0, \infty)$. For $t>1$,

$$
\frac{|x(t)|+1}{t+t\,|H(t)|} \leqq \frac{c_1}{t}+c_2+1+(k/\eta^2)\left[\int_0^t c_2\,|a(s)|\,s\,ds+\int_0^t |a(s)|\,(c_1+1)\,ds\right]
$$

and the result follows from (*).

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