

ENCLOSURE THEOREMS FOR EIGENVALUES OF ELLIPTIC OPERATORS

C. A. SWANSON¹

1. **Introduction.** Eigenvalue problems will be considered for linear, elliptic, self-adjoint partial differential operators on n -dimensional Euclidean space E^n . A typical point in E^n will be denoted by $x = (x^1, x^2, \dots, x^n)$, and the Euclidean norm of x by $|x|$. Partial differentiation with respect to x^i will be denoted by $D_i, i = 1, 2, \dots, n$. Elliptic operators L defined by

$$(1.1) \quad Lu = \frac{1}{c} \left\{ - \sum_{i,j=1}^n D_i(a_{ij}D_ju) + bu \right\}, \quad a_{ij} = a_{ji}$$

are to be considered when the coefficients a_{ij} , b , and c are continuous real-valued functions with $b \geq 0$, $c > 0$ in E^n . The ellipticity of L implies that the symmetric matrix (a_{ij}) is everywhere positive definite. A "solution" u of $Lu = 0$ is supposed to be of class C^1 and all derivatives involved in (1.1) are supposed to exist, be continuous, and satisfy $Lu = 0$ at every point.

The eigenvalue problem for L on E^n will be called the *basic problem*. The only assumption to be made is that there exists at least one eigenvalue λ for this problem whose associated eigenfunctions are " L -strongly asymptotic to zero" as $x \rightarrow \infty$ (definition in §2). Our purpose is to obtain variational formulae for the eigenvalues and eigenfunctions of L when E^n is perturbed to an n -disk of large radius a , and the null boundary condition is adjoined on the bounding $(n-1)$ -sphere. If the eigenspace of λ is m -dimensional, our first theorem shows in particular that at least m eigenvalues of the perturbed problem converge to λ as $a \rightarrow \infty$. Our other results are refinements of this which lead to asymptotic estimates for eigenfunctions. The method of estimation used here is due to H. F. Bohnenblust.

The problem at hand of estimating eigenvalues and eigenfunctions for large domains has its physical origin in certain models of enclosed quantum mechanical systems, considered by a number of authors including de Groot and ten Seldam [2], [12], Dingle [3], Hull and Julius [6], Sommerfeld and Hartman [7]. In the case that the Schrödinger operator (a special case of (1.1)) is separable, the problem reduces to a domain-perturbation problem for a singular second-

Received by the editors January 26, 1965.

¹ This research was supported by the United States Air Force Office of Scientific Research, under grant AF-AFOSR-379-63.

order ordinary differential operator, for which various results have been obtained [8], [9], [10].

2. Basic and perturbed problems. Let R_a denote the n -disk $\{x: |x| < a, a > 0\}$, and let $B = B_a$ denote the bounding $(n-1)$ -sphere. Let $\mathfrak{H}, \mathfrak{H}_a$ be the Hilbert spaces which are the Lebesgue spaces with respective inner products defined by

$$\langle u, v \rangle = \int_{E^n} u(x) \bar{v}(x) c(x) dx, \quad \langle u, v \rangle_a = \int_{R_a} u(x) \bar{v}(x) c(x) dx.$$

The Hilbert norms will be denoted as usual by $\|u\|, \|u\|_a$.

The *basic eigenvalue problem* for L is

$$(2.1) \quad Lu = \lambda u, \quad u \in \mathfrak{H},$$

where any eigenfunction is a solution of the differential equation in the sense described in §1.

The *perturbed domain* \mathfrak{D}_a is defined as the set of all complex-valued functions v which satisfy the following conditions:

- (i) v is continuous on \bar{R}_a ;
- (ii) v has uniformly continuous first partial derivatives in R_a ;
- (iii) all derivatives of v involved in Lv exist and are continuous in R_a ;
- (iv) v vanishes on B_a .

The perturbed eigenvalue problem under consideration is

$$(2.2) \quad Lv = \mu v, \quad v \in \mathfrak{D}_a.$$

It is known [1], [5] on account of the ellipticity of L on \bar{R}_a that there exists a denumerable set of eigenvalues $\{\mu_j\}$ ($0 < \mu_1 \leq \mu_2 \leq \dots$) and a corresponding complete orthonormal sequence of eigenfunctions v_j . Green's function $K_a(x, y)$ is constructed in the usual way as the sum of a fundamental solution of $Lv=0$ and the solution of a suitable Dirichlet problem. If K_a is the linear integral operator whose kernel is Green's function, the eigenfunctions v_j satisfy the integral equation

$$(2.3) \quad v_j = \mu_j K_a v_j, \quad j = 1, 2, \dots,$$

and any basic eigenfunction u satisfies $LK_a u = u$ in R_a [1].

A *parametrix* is a nondecreasing continuous function ϕ in $0 < a < \infty$ such that $\lim \phi(a) = \infty$ ($a \rightarrow \infty$). An *L-indicator* is a solution G of the differential equation $LG=0$ in E^n which is uniformly asymptotic to some parametrix ϕ as $x \rightarrow \infty$, i.e. $G(x) \sim \phi(|x|)$ uniformly in E^n . A function u is said to be *L-strongly asymptotic to zero* if there exists an *L-indicator* such that

$$(2.4) \quad \lim_{|x| \rightarrow \infty} \|u(x)\| G \|G(x)\| u \|_{|x|}^{-1} = 0$$

uniformly in E^n .

It is not true in general that the eigenvalues $\mu = \mu(a)$ of (2.2) tend to limits as $a \rightarrow \infty$, even when the spectrum of the basic problem is entirely discrete. Easy counterexamples are provided in the case $n = 1$ when the singularity at ∞ is of the limit circle type in Weyl's classification [9]. Here we shall prove the convergence of the eigenvalues $\mu(a)$ to basic eigenvalues under the assumption that all basic eigenfunctions are L -strongly asymptotic to zero.

As an example of (1.1), consider the Schrödinger operator $L = -\Delta + b$, where $b(x) = |x|^2 + o(1)$ as $|x| \rightarrow \infty$. Since $b(x) \rightarrow \infty$, the whole basic spectrum is discrete [13, p. 150]. A parametrix is $\phi(a) = a^{-n/2} \exp(a^2/2)$, $a > \sqrt{n}/2$, and every basic eigenfunction satisfies (2.4).

3. Asymptotic estimates for eigenvalues. The eigenspace associated with a basic eigenvalue λ will be denoted by \mathfrak{A}_λ . The following notations will be used

$$(3.1) \quad \begin{aligned} \psi_a[u] &= 2 \left(\max_B |u| \right) \|G\|_a(\phi(a)\|u\|_a)^{-1} \quad (u \neq 0) \\ \psi_a &= \sup_{u \in \mathfrak{A}_\lambda} \psi_a[u]; \quad \rho_a = 2\lambda\psi_a/(1 - 2\psi_a). \end{aligned}$$

Since every $u \in \mathfrak{A}_\lambda$ has the form $u = \sum_{i=1}^m \alpha_i u_i$ in terms of an orthonormal basis $\{u_i\}$, it is easy to verify that

$$\psi_a \leq 2m \max_{1 \leq i \leq m} \psi_a[u_i].$$

It follows from (2.4) that $\psi_a = o(1)$ and $\rho_a = o(1)$ as $a \rightarrow \infty$.

THEOREM 1. *If λ is a basic eigenvalue possessing m orthonormal eigenfunctions which are L -strongly asymptotic to zero, there exists a positive number a_0 such that at least m perturbed eigenvalues $\mu_i(a)$ of (2.2) are enclosed in the interval $[\lambda, \lambda + \rho_a]$ whenever $a \geq a_0$.*

PROOF. It follows from the maximum principle for elliptic differential equations [1, p. 326] that $\lambda > 0$. Let $\alpha = 1/\lambda$. For every $u \in \mathfrak{A}_\lambda$, the function $f = K_a u - \alpha u$ is the solution of the Dirichlet problem $Lf = 0$ in R_a , $f = -\alpha u$ on B_a . Define

$$(3.2) \quad g(x) = G(x)/\phi(a); \quad F(x) = 2 \left(\max_B |f| \right) g(x) - f(x).$$

Since $G(x) \sim \phi(|x|)$ as $x \rightarrow \infty$, there exists a positive number a_0 such that $g(x) \geq \frac{1}{2}$ on B_a and $2\psi_a < 1$ whenever $a \geq a_0$. Hence $F(x) \geq 0$ on B_a , and it follows from the maximum principle that $F(x) \geq 0$ throughout R_a , or

$$f(x) \leq 2 \left(\max_B |f| \right) g(x), \quad x \in R_a.$$

Similarly

$$f(x) \geq -2 \left(\max_B |f| \right) g(x), \quad x \in R_a,$$

and consequently

$$\|f\|_a \leq 2\alpha \left(\max_B |u| \right) \|G\|_a / \phi(a),$$

or by (3.1),

$$(3.3) \quad \|K_a u - \alpha u\|_a \leq 2\alpha\psi_a \|u\|_a.$$

Let $P(\epsilon)$ be the projection operator from \mathfrak{H}_a onto the subspace $\mathfrak{H}_{a\epsilon}$ spanned by all eigenfunctions of K_a whose corresponding eigenvalues β_i lie in the interval $|\beta - \alpha| < \epsilon$. The following inequality is valid for arbitrary $u \in \mathfrak{H}_a$

$$\|u - P(\epsilon)u\|_a \leq \epsilon^{-1} \|K_a u - \alpha u\|_a.$$

The proof given in [11] for self-adjoint transformations extends without change to K_a . Then (3.3) yields the inequality

$$(3.4) \quad \|u - P(\epsilon)u\|_a \leq 2\alpha\psi_a \epsilon^{-1} \|u\|_a,$$

which implies that at least m eigenvalues β_i of K_a are included in the interval $|\beta_i - \alpha| \leq 2\alpha\psi_a$, $i = 1, 2, \dots$ [11, p. 35]. Since α, β_i are reciprocals of λ, μ_i respectively, at least m eigenvalues μ_i of the perturbed problem (2.2) satisfy $|\mu_i - \lambda| \leq 2\mu_i\psi_a$. Since $\mu_i \geq \lambda$ is a general consequence of the minimax principle for eigenvalues [1], $\lambda \leq \mu_i \leq \lambda + 2\mu_i\psi_a$, or $\lambda \leq \mu_i \leq \lambda/(1 - 2\psi_a) = \lambda + \rho_a$, where ρ_a is defined by (3.1). This completes the proof of Theorem 1.

THEOREM 2. *Let λ be a basic eigenvalue of multiplicity m whose eigenfunctions are all L -strongly asymptotic to zero. If there exists a basic eigenvalue exceeding λ , then there is a positive number a_1 such that exactly m perturbed eigenvalues μ_i are enclosed in the interval $[\lambda, \lambda + \rho_a]$ whenever $a \geq a_1$.*

PROOF. Suppose first that λ is the smallest basic eigenvalue. Let λ' be the smallest eigenvalue exceeding λ . Since $\psi_a = o(1)$ as $a \rightarrow \infty$, there is a number $a_1 \geq a_0$ such that $\psi_a < (\lambda' - \lambda)/2\lambda'$ whenever $a \geq a_1$, which implies $\lambda + \rho_a < \lambda'$. Then theorem 1 shows that at least m eigenvalues μ_i are included in the subinterval $[\lambda, \lambda + \rho_a]$ of $[\lambda, \lambda']$. Since $\mu_i \geq \lambda_i$ for each i is a general consequence of the minimax property of eigenvalues, at most m eigenvalues μ_i lie in this subinterval, and hence exactly m . If $\lambda = \lambda^i$ is the i th *distinct* basic eigenvalue, $\lambda^1 < \lambda^2 < \dots$, an easy induction proof establishes the same result.

4. Uniform estimates for eigenfunctions. Let $p = p(n)$ be a positive number satisfying $p(2) = 0$, $p(3) = 0$, and $0 < n - 2p < 4$. Because the fundamental singularity of $K_a(x, y)$ is of order $|x - y|^{2-n}$, $n \geq 3$, the function

$$k_a(x) = \left(\int_{R_a} |x - y|^{2p} K_a^2(x, y) c(y) dy \right)^{1/2}$$

is well-defined in R_a . Our assumption for the next theorem is that

$$(4.1) \quad \psi_a^q k_a(x) = o(1) \quad \text{as } a \rightarrow \infty \quad (q = (n - 2p)/n)$$

uniformly in R_a , where ψ_a is defined by (3.1). In the case $n = 1$, $p = 0$ considered in [10, p. 310], $k_a(x)$ is uniformly bounded in R_a for $a \geq a_0$, and accordingly (4.1) is implied by (2.4).

THEOREM 3. *Corresponding to the eigenvalues λ and μ_i of Theorem 2, there are orthonormal eigenfunctions u_i associated with λ and v_i associated with the μ_i such that*

$$(4.2) \quad \begin{aligned} v_i(x) &= u_i(x) - f_i(x) + O(\psi_a^q) k_a(x), \\ i &= 1, 2, \dots, m; \quad x \in R_a; \quad a \geq a_1, \end{aligned}$$

where f_i is the solution of the Dirichlet problem $Lf_i = 0$ in R_a , $f_i = u_i$ on B_a .

PROOF. Select the number ϵ in (3.4) to be $\alpha - \alpha'$, where $\alpha = 1/\lambda$, $\alpha' = 1/\lambda'$. With a_1 as in Theorem 2, it follows that $2\alpha\psi_a < \alpha(\lambda' - \lambda)/\lambda' = \alpha - \alpha' = \epsilon$ for $a \geq a_1$. Then $\mathfrak{F}_{a\epsilon}$ is m -dimensional by Theorem 2 and $P(\epsilon)u = 0$ implies $u = 0$ by (3.4). Hence there exist m uniquely determined linearly independent eigenfunctions z_i corresponding to α which $P(\epsilon)$ maps into the orthonormal eigenfunctions v_i , and by (3.4), $\|z_i - v_i\|_a = O(\psi_a)$. Since

$$|\langle z_i, z_j \rangle_a - \langle v_i, v_j \rangle_a| \leq \|v_i\|_a \|z_j - v_j\|_a + \|z_j\|_a \|z_i - v_i\|_a$$

by the Schwarz inequality, $\langle z_i, z_j \rangle_a = \delta_{ij} + O(\psi_a)$, $i, j = 1, 2, \dots, m$. Since the z_i are linearly independent, an orthonormal sequence $\{u_i\}$ can be constructed by the Schmidt process as linear combinations of the z_i , and it is seen without difficulty that $\|u_i - z_i\|_a = O(\psi_a)$. Hence

$$(4.3) \quad \|u_i - v_i\|_a = O(\psi_a), \quad i = 1, 2, \dots, m.$$

Omitting the subscripts i , we select a typical u in the set $\{u_i\}$ and corresponding v in the set $\{v_i\}$. Since $\mu - \lambda = O(\psi_a)$ by Theorem 2 and $\|v - u\|_a = O(\psi_a)$ by (4.3), we obtain

$$\|\mu v - \lambda u\|_a \leq \mu \|v - u\|_a + (\mu - \lambda) \|u\|_a = O(\psi_a).$$

Define

$$w_a(x) = \left(\int_{R_a} |x - y|^{-2p} |\mu v(y) - \lambda u(y)|^2 c(y) dy \right)^{1/2}.$$

Let $S(x, \delta)$ denote the n -disk of centre x and radius δ . A routine decomposition of the integral into integrals over $S(x, \delta) \cap R_a$ and the remainder of R_a yields

$$w_a^2(x) \leq \delta^{-2p} \|\mu v - \lambda u\|_a^2 + O(\delta^{n-2p}).$$

With the choice $\delta = \psi_a^{2/n}$ we obtain the uniform estimate $w_a(x) = O(\psi_a^q)$, where $0 < q = (n - 2p)/n < 4/n$. In particular, $w_a(x) = O(\psi_a)$ if $n = 2$ or 3 . Since

$$v(x) - \lambda K_a u(x) = K_a(\mu v - \lambda u)(x),$$

it follows from Schwarz's inequality that

$$(4.4) \quad |v(x) - \lambda K_a u(x)| \leq k_a(x) w_a(x) = O(\psi_a^q) k_a(x).$$

The function g defined by

$$(4.5) \quad g(x) = \lambda K_a u(x) - u(x) + f(x)$$

is a solution of the Dirichlet problem $Lg = 0$ in R_a , $g = 0$ on B_a , and hence g is identically zero. The uniform estimate (4.2) is then a consequence of (4.4) and (4.5).

5. Asymptotic variational formulae for eigenvalues. We shall require Green's symmetric identity in the form [4]

$$(5.1) \quad \langle Lu, v \rangle_a - \langle u, Lv \rangle_a = [uv]_a - [vu]_a = \{uv\}_a,$$

where

$$[uv]_a = \int_{B_a} u \sum_{k,j=1}^n a_{kj} n_k D_j \bar{v} dS$$

and $\{uv\}_a$ is defined by (5.1). Here (n_k) is the outward pointing unit normal to B_a .

Let u, v be normalized eigenfunctions associated with λ, μ , as described in Theorems 2 and 3. Let f be the solution of the Dirichlet problem $Lf=0$ in R_a , $f=u$ on B_a .

Since $v=0$ on B_a , $[vu]_a=0$. Then application of (5.1) to the differential equations $Lu=\lambda u$ and $Lv=\mu v$ leads to the formula

$$(5.2) \quad (\lambda - \mu)\langle u, v \rangle_a = [uv]_a.$$

It is a consequence of (4.3) that

$$|\langle u, v \rangle_a - \langle u, u \rangle_a| \leq \|u\|_a \|v - u\|_a = O(\psi_a).$$

Hence $\langle u, v \rangle_a = 1 + O(\psi_a)$ and (5.2) yields

$$(5.3) \quad \lambda - \mu = [uv]_a [1 + O(\psi_a)].$$

Application of (5.1) to the differential equations $Lu=\lambda u$, $Lv=\mu v$, and $Lf=0$ leads to

$$(5.4) \quad -\mu\langle f, v \rangle_a = [fv]_a = [uv]_a;$$

$$(5.5) \quad -\lambda\langle f, u \rangle_a = \{fu\}_a.$$

Since $\mu=\lambda+O(\psi_a)$ by Theorem 2, it follows from (5.3) and (5.4) that

$$\lambda - \mu = -\lambda\langle f, v \rangle_a [1 + O(\psi_a)].$$

Finally we appeal to the uniform estimate (4.2) and to (5.5) to obtain

$$\lambda - \mu = [\{fu\}_a - \lambda\langle f, f \rangle_a][1 + O(\psi_a)] + \langle f, k_a \rangle_a O(\psi_a^2).$$

In some cases the first term dominates the other terms, and we obtain the asymptotic form

$$(5.6) \quad \mu(a) - \lambda \sim \{uf\}_a \quad \text{as } a \rightarrow \infty.$$

The results of Theorems 1-3 are then sharpened accordingly.

In the example considered at the end of §2, some of the basic eigenfunctions are asymptotic to radial functions $R(|x|)$ (explicit formulae in [13]). In such cases, $f(x) \sim R(a)\phi(|x|)/\phi(a)$, and (5.6) yields the eigenvalue variation $\mu(a) - \lambda \sim \omega_{n-1}a^n |R(a)|^2$, where ω_{n-1} denotes the volume of the unit $(n-1)$ -sphere.

REFERENCES

1. R. Courant and D. Hilbert, *Methods of mathematical physics*, Vol. II, Interscience, New York, 1962.
2. S. R. de Groot and C. A. Seldam, *On the energy levels of a model of the compressed hydrogen atom*, Physica 12 (1946), 669–682.
3. R. B. Dingle, *The solution of the Schrödinger equation for finite systems, with special reference to the motion of electrons in Coulomb electric fields and uniform magnetic fields*, Proc. Cambridge, Philos. Soc. 49 (1953), 102–114.
4. G. F. D. Duff, *Partial differential equations*, Univ. Toronto Press, Toronto, 1956.
5. Lars Gårding, *On the asymptotic distribution of the eigenvalues and eigenfunctions of elliptic differential operators*, Math. Scand. 1 (1953), 237–255.
6. T. E. Hull and R. S. Julius, *Enclosed quantum mechanical systems*, Canad. J. Phys. 34 (1956), 914–919.
7. A. Sommerfeld and H. Hartman, *Künstliche Grenzbedingungen in der Wellenmechanik. Der beschränkte Rotator*, Ann. Phys. (5) 37 (1940), 333–343.
8. C. A. Swanson, *Differential operators with perturbed domains*, J. Rational Mech. Anal. 6 (1957), 823–846.
9. ———, *Asymptotic estimates for limit circle problems*, Pacific J. Math. 11 (1961), 1549–1559.
10. ———, *Asymptotic estimates for limit point problems*, Pacific J. Math. 13 (1963), 305–316.
11. ———, *On spectral estimation*, Bull. Amer. Math. Soc. 68 (1962), 33–35.
12. C. A. ten Seldam and S. R. de Groot, *On the ground state of a model for compressed helium*, Physica 18 (1952), 891–904.
13. E. C. Titchmarsh, *Eigenfunction expansions associated with second order differential equations*, Part II, Oxford Univ. Press, Oxford, 1958.

THE UNIVERSITY OF BRITISH COLUMBIA