WEAK COMPACTNESS IN LOCALLY CONVEX SPACES

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1. **Introduction.** A recently published paper of R. C. James [1] proves the following Theorem: A weakly closed set C in a Banach space B is weakly compact if and only if every bounded linear functional on B attains its supremum on C at some point of C. The proof given by James is rather long and involved: the following, while not employing any basically different ideas, is a simpler version and extends the theorem with no extra effort to deal with a locally convex linear topological space rather than a Banach space, using the Eberlein criterion for weak compactness (see e.g. [2, p. 159]).

2. **The result.**

THEOREM. *Let C be a weakly closed bounded subset of the real and complete locally convex linear topological space E. Then C is weakly compact if and only if given any element f of the dual E^* of E, there is $x \in C$ such that $f(x) = \sup\{f(u) : u \in C\}$.*

COROLLARY. *The hypothesis that E be complete may be replaced by the hypothesis that the closed convex hull of C be complete (in the original topology of E).*

PROOF. The implication one way is elementary: namely, suppose C is weakly compact and f any element of E^*. Then by the definition of the weak topology f is continuous on C in the weak topology and so attains its bounds.

We prove the implication the other way by assuming that C is not weakly compact, and constructing a continuous linear functional which does not attain its supremum on C at any point of C. The proof of this fact is divided up into a series of lemmas.

LEMMA 1. *There is a sequence (z_n) of points in C and a sequence (f_n) of elements of E^* such that $\{f_n\}$ is an equicontinuous set and the limits $\lim_i \lim_j f_i(z_j)$ and $\lim_j \lim_i f_i(z_j)$ exist and are unequal.*

For the proof of this result, which is Eberlein's celebrated compactness theorem, see [2], where the result is stated on p. 159.

We now introduce some notation. Since we shall not be dealing only with functionals on E that are linear, we denote by F the set of all real-valued continuous functions on E which are positive-homogeneous,

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$$f(\alpha x) = \alpha f(x) \qquad (\alpha \geqq 0).$$

Since for each $f \in F$, there is a neighbourhood U of 0 in E such that $|f(x)| = |f(x) - f(0)| \leqq 1$ $(x \in U)$, elements of F are bounded on bounded sets. Note that E^* is a subspace of F. We give F the weak* topology $w(F, E)$ a pointwise convergence on E; this makes E^* a weak*-closed subspace.

We define

$$p(f) = \sup\{f(x) : x \in C\} \qquad (f \in F).$$

The functional p is finite-valued and has the following properties:

(i) p is sublinear, i.e. $p(\lambda f) = \lambda p(f)$ for $\lambda \geqq 0$ and

$$p(f + g) \leqq p(f) + p(g).$$

(ii) Since

$$p(f) \leqq p(g) + p(f - g)$$

and

$$p(g) \leqq p(f) + p(g - f)$$

we have

$$-p(g - f) \leqq p(f) - p(g) \leqq p(f - g).$$

(iii) If $A \subseteq F$ and A is equicontinuous then p is bounded on A, for there is a neighbourhood U of 0 in E such that $|f(x)| \leqq 1$ for all $x \in U$, $f \in A$, and C is absorbed by U.

We also define $P(f) = \sup\{|f(x)| : x \in C\}$ for $f \in F$. The functional P is a seminorm inducing on F the topology of uniform convergence on C.

Let (f_i) be a sequence in F which is equicontinuous at each point of E, and define functions $G_- = \liminf f_i$, $G^- = \limsup f_i$ by

$$G_-(x) = \liminf f_i(x), \qquad G^-(x) = \limsup f_i(x) \qquad (x \in E).$$

Given x_0 and ϵ there is a neighbourhood U of x_0 such that

$$\sup\{|f_i(x) - f_i(x_0)| : i = 1, 2, \cdots; x \in U\} \leqq \epsilon.$$

Applying this and the relation $|\liminf f_i(x) - \liminf f_i(x_0)| \leqq \sup_i |f_i(x) - f_i(x_0)|$ first to 0 and then to an arbitrary x_0 we see that G_-, and similarly G^-, is everywhere finite and continuous. It is clearly positive-homogeneous and so belongs to F.

LEMMA 2. *Let (f_i) be a sequence in F equicontinuous at each point. Let the topology on F be that of the seminorm P, and let X be any subset of F which is separable in the relative topology. Then there is a subse-*

quence (G_i) *of the* f_i *such that if* $G_- = \liminf G_i$, $G^- = \limsup G_i$, *we have* $p(f-G_-) = p(f-G^-)$ *for all* $f \in X$.

PROOF. Let (ω_k) be a dense sequence in X. By replacing it by the sequence $\omega_1, \omega_1, \omega_2, \omega_1, \omega_2, \omega_3, \omega_1, \omega_2, \omega_3, \omega_4, \cdots$, we can assume that each point of X is a cluster point of the sequence. We now apply a diagonal process, inductively defining points x_n and sequences $(f_i^n : i = 1, 2, \cdots)$ as follows:

For $n=1$ choose $x_1 \in C$ so that

$$\omega_1(x_1) - \liminf f_i(x_1) > p(\omega_1 - \liminf f_i) - \tfrac{1}{2},$$

while for $n>1$ choose $x_n \in C$ so that

$$\omega_n(x_n) - \liminf_i f_i^{n-1}(x_n) > p\left(\omega_n - \liminf_i f_i^{n-1}\right) - 2^{-n},$$

and $(f_i^n : i=1, 2, \cdots)$ as a subsequence of $(f_i^{n-1} : i=2, 3, \cdots)$, so that

$$f_i^n(x_n) \text{ converges to } \liminf_i f_i^{n-1}(x_n) \quad \text{as } i \to \infty.$$

(Note that f_1^{n-1} thus does not occur as a member of (f_i^n).) Now define $G_k = f_1^k$. Since $(G_n, G_{n+1}, \cdots)$ is for each n a subsequence of $(f_1^n, f_2^n, \cdots)$ we have, if G_-, G^- denote $\liminf G_k$, $\limsup G_k$, for every n,

(i) $$\lim_k G_k(x_n) \text{ exists and equals } \lim_i f_i^n(x_n);$$

(ii) $$\begin{aligned} \omega_n(x_n) - G_-(x_n) &= \omega_n(x_n) - \lim_i f_i^n(x_n) = \omega_n(x_n) - \liminf_i f_i^{n-1}(x_n) \\ &> p\left(\omega_n - \liminf_i f_i^{n-1}\right) - 2^{-n} \\ &\geqq p\left(\omega_n - \liminf_k G_k\right) - 2^{-n} = p(\omega_n - G_-) - 2^{-n}. \end{aligned}$$

Now let f be an element of X. Because of the cluster point property of the ω_k, given any $\epsilon > 0$ there is n such that (i) $2^{-n} < \epsilon$ and (ii) for all x in C, $|f(x) - \omega_n(x)| < \epsilon$. Then we have

$$\begin{aligned} p(f - G_-) &\leqq \epsilon + p(\omega_n - G_-) \\ &< 2\epsilon + \omega_n(x_n) - G_-(x_n) \\ &= 2\epsilon + \omega_n(x_n) - G^-(x_n) \\ &< 3\epsilon + f(x_n) - G^-(x_n) \\ &\leqq 3\epsilon + p(f - G^-). \end{aligned}$$

Since ϵ is arbitrary we have $p(f-G_-) \leqq p(f-G^-)$; the opposite inequality is trivial and hence $p(f-G_-) = p(f-G^-)$. This proves the lemma.

Let (f_i) now be the sequence of Lemma 1 and X be the linear span of the f_i. In the P topology X is separable (e.g. take linear combinations of the f_i with rational coefficients), so the conditions of Lemma 2 are satisfied, and we can by taking a subsequence assume that

$$p(f - G_-) = p(f - G^-) \qquad (f \in X),$$

where $G_- = \lim\inf f_i$, $G^- = \lim\sup f_i$. The double limit relation of Lemma 1 is not disturbed by this process. Further we can without loss of generality assume that $f_k(z_j) - \lim_i f_i(z_j)$ is for each k eventually $\geqq r > 0$, as j tends to infinity, (by another application of the diagonal process).

Let K_n denote the convex hull of $\{f_n, f_{n+1}, \cdots\}$ for $n = 1, 2, \cdots$. To keep the record straight, we have $F \supseteq E^* \supseteq X \supseteq K_1 \supseteq K_2 \supseteq \cdots$.

LEMMA 3. *For all $f \in K_1$, $p(f-G_-) \geqq r$.*

PROOF. Let f be any element of K_1; then $f = \sum_{i=1}^s \lambda_i f_{n_i}$, where $\lambda_i \geqq 0$, and $\sum_1^s \lambda_i = 1$. Then

$$\begin{aligned} p(f - G_-) \geqq f(z_j) - G_-(z_j) &= \sum_1^s \lambda_i \{f_{n_i}(z_j) - G_-(z_j)\} \\ &= \sum_1^s \lambda_i \left\{ f_{n_i}(z_j) - \lim_r f_r(z_j) \right\} \\ &\geqq \sum_1^s \lambda_i r = r \end{aligned}$$

if we choose j large enough.

LEMMA 4. *Let Y be a linear space, and ρ, β, β' be strictly positive numbers. Let A be a convex subset of Y, u a point of Y, and p a sublinear functional on Y. Suppose that*

$$\inf_{a \in A} p(u + \beta a) > \beta\rho + p(u).$$

Then there is a point a_0 in A such that

$$\inf_{b \in A} p(u + \beta a_0 + \beta' b) > \beta'\rho + p(u + \beta a_0).$$

PROOF. Choose any x, y in A and set $c = (\beta x + \beta' y)/(\beta + \beta')$. Then $c \in A$ and $u + \beta x + \beta' y = u + (\beta + \beta')c = (1 + \beta'/\beta)(u + \beta c) - (\beta'/\beta)u$. From the hypothesis of the lemma,

$$-p(u) = \beta\rho - \inf_{a \in A} p(u + \beta a) + \delta \qquad (\delta > 0)$$

and by sublinearity,

$$p(u+\beta x+\beta' y) \geqq p((1+\beta'/\beta)(u+\beta c)) - p((\beta'/\beta)u).$$

Hence for fixed a_0 in A,

$$\begin{aligned}
&\inf_{b\in A} p(u+\beta a_0+\beta' b)\\
&\geqq \left(1+\frac{\beta'}{\beta}\right)\inf\left\{p(u+\beta c): c=\frac{\beta a_0+\beta' b}{\beta+\beta'},\, b\in A\right\} - \frac{\beta'}{\beta}p(u)\\
&\geqq \left(1+\frac{\beta'}{\beta}\right)\inf_{a\in A} p(u+\beta a) - \frac{\beta'}{\beta}p(u)\\
&= \left(1+\frac{\beta'}{\beta}\right)\inf_{a\in A} p(u+\beta a) + \frac{\beta'}{\beta}\left(\beta\rho - \inf_{a\in A} p(u+\beta a)\right) + \frac{\beta'}{\beta}\delta\\
&= \beta'\rho + \inf_{a\in A} p(u+\beta a) + \frac{\beta'}{\beta}\delta.
\end{aligned}$$

Thus if we choose a_0 so that $p(u+\beta a_0) < \inf_{a\in A} p(u+\beta a) + (\beta'/\beta)\delta$ we obtain the required result.

LEMMA 5. *Let (β_n) be an arbitrary sequence of strictly positive real numbers. Then there is a sequence (g_n) in F such that for all n, $g_n \in K_n$ and*

$$p\left[\sum_1^n \beta_i(g_i - G_-)\right] > \frac{1}{2}\beta_n r + p\left[\sum_1^{n-1}\beta_i(g_i - G_-)\right].$$

PROOF. We use induction and Lemma 4.

For the first step, let $u=0$, $\beta=\beta_1$, $\beta'=\beta_2$ and A be the set $K_1 - G_- = \{f - G_-: f\in K_1\}$; and p as already defined. Then $\inf_{f\in A} p(u+\beta f) = \inf_{f\in K_1} p[\beta_1(f-G_-)] \geqq \beta_1 r > \frac{1}{2}\beta_1 r + p(u)$ by Lemma 3, so the conditions of Lemma 4 are satisfied. Hence there is $g_1\in K_1$ such that

$$\inf_{g\in K_1} p[\beta_1(g_1-G_-)+\beta_2(g-G_-)] > \tfrac{1}{2}\beta_2 r + p[\beta_1(g_1-G_-)].$$

For the nth step, let $u=\sum_1^{n-1}\beta_i(g_i-G_-)$, $\beta=\beta_n$, $\beta'=\beta_{n+1}$ and A be the set $K_n - G_-$. By the inductive hypothesis, and since $K_{n-1}\supseteq K_n$,

$$\inf_{f\in A} p(u+\beta f) \geqq \inf\{p(u+\beta f): f\in K_{n-1}-G_-\} > \tfrac{1}{2}\beta_n r + p(u)$$

and Lemma 4 gives $g_n\in K_n$ such that if $v=\sum_1^n \beta_i(g_i-G_-)$,

$$\inf_{f\in A} p(v+\beta' f) > \tfrac{1}{2}\beta' r + p(v)$$

which is the inductive hypothesis for n. The sequence (g_n) then has the required property.

LEMMA 6. *There is G_0 in E^* such that*
(i) $\liminf g_n(x) \leqq G_0(x) \quad (x \in E)$,
(ii) $p(h-G_0) = p(h-G_-) \quad (h \in X)$.

PROOF. The set K_1 is the convex hull of the equicontinuous sequence (f_n), and thus the weak*-closure of K_1 in E^* is weak*-compact. The sequence (g_n) lies in K_1 and therefore has a weak* cluster-point G_0 in E^*. Then for each x in E, $G_0(x)$ is a cluster-point of the real number sequence $(g_n(x))$, and so

$$\liminf g_n(x) \leqq G_0(x) \leqq \limsup g_n(x),$$

which establishes (i).

Next, since $g_n \in K_n$, $g_n(x)$ is a convex combination $\sum \lambda_i f_{m_i}(x)$, with the m_i not less than n. It follows that there is at least one of the m_i for which $f_{m_i}(x) \leqq g_n(x)$. In other words, given any n there is $m \geqq n$ such that

$$f_m(x) \leqq g_n(x),$$

and so $G_-(x) = \liminf f_n(x) \leqq \limsup g_n(x)$.

A similar argument on the other side establishes $G^-(x) \geqq \limsup g_n(x)$. Combining our inequalities we have $G_-(x) \leqq G_0(x) \leqq G^-(x)$, $(x \in E)$ and so

$$p(h - G_-) \geqq p(h - G_0) \geqq p(h - G^-) \qquad (h \in X).$$

The outer terms are equal and the lemma is proved.

COROLLARY. *The conclusion of Lemma 5 holds with G_- replaced by G_0.*

PROOF. Fix n and let $\alpha = \beta_1 + \cdots + \beta_n$. Then

$$p\left[\sum_1^n \beta_i(g_i - G_0)\right] = \alpha p\left[\frac{1}{\alpha}\sum_1^n \beta_i g_i - G_0\right] = \alpha p\left[\frac{1}{\alpha}\sum_1^n \beta_i g_i - G_-\right]$$
$$= p\left[\sum_1^n \beta_i(g_i - G_-)\right],$$

and the result is now clear.

LEMMA 7. *If the sequence (β_n) decreases to zero fast enough (more precisely if $(\sum_{n+1}^\infty \beta_i)/\beta_n \to 0$ as $n \to \infty$), the series*

$$\sum_{i=1}^\infty \beta_i(g_i - G_0)$$

defines an element g of E^* which does not attain a maximum on C.

Proof. Assume to start with only that $\sum \beta_i$ converges. Now K_1, hence also $K_1 - G_0$, is equicontinuous; hence there is a neighbourhood U of 0 in E such that

$$x \in U \Rightarrow |f(x)| \leqq 1 \qquad \text{for all } f \in K_1 - G_0.$$

Hence

$$x \in U \Rightarrow \sum_{i=1}^{\infty} \beta_i [g_i(x) - G_0(x)] \leqq \sum_{i=1}^{\infty} \beta_i.$$

This shows that g is defined and continuous on E, i.e. $g \in E^*$. Now by the note (iii) after the definition of p, there is $M \geqq 0$ such that

$$x \in C, \qquad f \in K_1 - G_0 \Rightarrow |f(x)| \leqq M.$$

Suppose that g attains its supremum on C at some point u of C. Then for each n, we have

$$\begin{aligned}
\sum_1^n \beta_i(g_i - G_0)(u) &= g(u) - \sum_{n+1}^{\infty} \beta_i (g_i - G_0)(u) \geqq g(u) - M \sum_{n+1}^{\infty} \beta_i \\
&= p(g) - M \sum_{n+1}^{\infty} \beta_i \geqq p\left[\sum_1^n \beta_i (g_i - G_0) \right] \\
&\quad - p\left[\sum_1^n \beta_i (g_i - G_0) - g \right] - M \sum_{n+1}^{\infty} \beta_i \\
&\geqq p\left[\sum_1^n \beta_i (g_i - G_0) \right] - 2M \sum_{n+1}^{\infty} \beta_i \\
&> \frac{1}{2} \beta_n r + p\left[\sum_1^{n-1} \beta_i (g_i - G_0) \right] - 2M \sum_{n+1}^{\infty} \beta_i
\end{aligned}$$

by Lemma 6 (Corollary)

$$\geqq \frac{1}{2} \beta_n r + \sum_1^{n-1} \beta_i (g_i - G_0)(u) - 2M \sum_{n+1}^{\infty} \beta_i.$$

Hence

$$(g_n - G_0)(u) > \frac{1}{2} r - 2M \left(\sum_{n+1}^{\infty} \beta_i \right) \Big/ \beta_n.$$

If we choose (β_n) to decrease fast enough, for instance $\beta_n = 1/n!$, we find that lim inf $(g_n - G_0)(u) \geqq \frac{1}{2} r$, which contradicts the fact that lim inf $g_n(u) \leqq G_0(u)$. Hence g cannot attain its supremum on C at any point of C, and the theorem is proved.

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