A PROPERTY OF l_p SPACES

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1. **Introduction.** In 1936 J. A. Clarkson [**1**, p. 396] introduced the notion of uniform convexity of the norm in a Banach space and showed [**1**, p. 403] that if $1<p<\infty$ then the space l_p is uniformly convex.

It is the object of this paper to consider a generalized type of uniform convexity, which we shall call weak uniform convexity. In §2 we prove that if $1 \leqq p \leqq \infty$ then l_p is weakly uniformly convex. In §3 we introduce the concepts of a norm interval and a norm convex set, and we prove a "nearest point" theorem for norm convex sets.

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DEFINITION 1.1. The statement that the Banach space S is uniformly convex means that if $\epsilon>0$ then there exists a $\delta>0$ such that if $\|x\|=\|y\|=1$ and $\|x-y\| \geqq \epsilon$, then $\|\frac{1}{2}x+\frac{1}{2}y\| \leqq 1-\delta$.

DEFINITION 1.2. The statement that the Banach space S is weakly uniformly convex means that if $\epsilon>0$ then there exists a $\delta>0$ such that if $\|x\|=\|y\|=1$ and $\|x-y\| \geqq \epsilon$, then there exists a point, w, such that $\|x-w\|+\|w-y\|=\|x-y\|$ and $\|w\| \leqq 1-\delta$.

2. **A property of l_p spaces.** Recall that l_p $(1 \leqq p < \infty)$ is defined to be the space of real number sequences $(x_1, x_2, \cdots)$ such that $\sum_{i=1}^{\infty} |x_i|^p$ converges, with norm $\|x\| = [\sum_{i=1}^{\infty} |x_i|^p]^{1/p}$, and that l_∞ is the space of bounded real number sequences with least upper bound norm. We note that if S is a uniformly convex Banach space then S is weakly uniformly convex. Hence if $1<p<\infty$ then l_p is weakly uniformly convex.

THEOREM 2.1. *The space l_∞ is weakly uniformly convex.*

PROOF. Suppose that $0<\epsilon \leqq 2$, and suppose that $x=(x_1, x_2, \cdots)$ and $y=(y_1, y_2, \cdots)$ are points of l_∞ such that $\|x\|=\|y\|=1$ and $\|x-y\| \geqq \epsilon$. Let $m=\frac{1}{2}\|x-y\|$ and let $r=(r_1, r_2, \cdots)$ be the point of l_∞ such that for each positive integer i, r_i is the number in the common part of $[x_i-m, x_i+m]$ and $[y_i-m, y_i+m]$ which is smallest in absolute value.

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Now for each positive integer i, $|x_i - r_i| \leqq m$ and $|y_i - r_i| \leqq m$. Thus $|x_i - r_i| + |r_i - y_i| \leqq \|x - y\|$, and hence $\|x - r\| + \|r - y\| = \|x - y\|$.

Also, if $x_i \leqq y_i$, then:

$$r_i = \begin{cases} y_i - m, & \text{if } y_i - m \geqq 0, \\ 0, & \text{if } y_i - m < 0 \leqq x_i + m, \\ x_i + m, & \text{if } x_i + m < 0. \end{cases}$$

If $y_i - m \geqq 0$, then $|r_i| = y_i - m \leqq 1 - m$, and if $x_i + m < 0$, then $|r_i| = |x_i + m| = -x_i - m \leqq 1 - m$.

By a similar argument if $y_i \leqq x_i$ then $|r_i| \leqq 1 - m$.

Thus $\|r\| \leqq 1 - \frac{1}{2}\|x - y\| \leqq 1 - \frac{1}{2}\epsilon$, and l_∞ is weakly uniformly convex.

THEOREM 2.2. *The space l_1 is weakly uniformly convex.*

PROOF. Suppose that $0 < \epsilon \leqq 2$, and suppose that $x = (x_1, x_2, \cdots)$ and $y = (y_1, y_2, \cdots)$ are points of l_1 such that $\|x\| = \|y\| = 1$ and $\|x - y\| \geqq \epsilon$. Let $r = (r_1, r_2, \cdots)$ be the point of l_1 defined by:

$$r_i = \begin{cases} 0, & \text{if } x_i y_i \leqq 0, \\ x_i, & \text{if } x_i y_i > 0 \text{ and } |x_i| \leqq |y_i|, \\ y_i, & \text{if } x_i y_i > 0 \text{ and } |y_i| < |x_i|. \end{cases}$$

Then by an argument similar to the proof of Theorem 2.1, we obtain the following equalities:

(1) $|x_i - r_i| + |r_i - y_i| = |x_i - y_i|$,

(2) $|x_i - r_i| = |x_i| - |r_i|$,

(3) $|r_i - y_i| = |y_i| - |r_i|$.

Thus $\|x - r\| + \|r - y\| = \|x - y\|$, and $\|r\| \leqq 1 - \frac{1}{2}\epsilon$. Hence l_1 is weakly uniformly convex.

3. **Some properties of norm convex sets.** In this section we suppose that S is a Banach space with origin N.

DEFINITION 3.1. Suppose that P and Q are points of S. Then $[P, Q]^*$ (called the norm interval from P to Q) is the point set

$$A = \{R \text{ in } S \mid \|P - R\| + \|R - Q\| = \|P - Q\|\}.$$

DEFINITION 3.2. The statment that the point set M is norm convex means that if P and Q are points of M then each point of $[P, Q]^*$ is in M.

THEOREM 3.1. *The following two statements are equivalent*:

(1) *there exist point P and Q such that $[P, Q]^* \neq [P, Q]$*,

(2) *there exist three points x, y, and w such that w is in $[x, y]$ and $\|x\| = \|y\| = \|w\| = 1$.*

PROOF. Suppose (1) is true, and suppose R is a point of $[P, Q]^*$ which is not in $[P, Q]$. Let:

$$t = \frac{\|R - Q\|}{\|P - Q\|} \text{ and let } M = tP + (1 - t)Q.$$

Then $\|P-M\| = \|P-R\|$.

Now if $Z = \frac{1}{2}R + \frac{1}{2}M$, then $\|P-Z\| \leqq \|P-R\|$ and $\|Z-Q\| \leqq \|R-Q\|$, and it follows that $\|P-Z\| = \|P-R\|$.

Hence, if

$$x = \frac{P - R}{\|P - R\|}, \qquad y = \frac{P - M}{\|P - R\|}, \quad \text{and} \quad w = \frac{P - Z}{\|P - R\|},$$

then $\|x\| = \|y\| = \|w\| = 1$. Since $w = \frac{1}{2}x + \frac{1}{2}y$, (2) is true.

Suppose (2) is true. Then it follows from the triangle inequality that if $r = \frac{1}{2}x + \frac{1}{2}y$ then $\|r\| = 1$. Thus:

$$\|(x + y) - N\| = \|(x + y) - x\| + \|x - N\|.$$

Hence x is in $[N, x+y]^*$ but x is not in $[N, x+y]$. Therefore (1) is true.

THEOREM 3.2. *The following two statements are equivalent*:

(1) *S is weakly uniformly convex*;

(2) *if $\epsilon > 0$ then there exists a $\delta > 0$ such that if $\|x\| = \|y\| = 1 + \delta$ and $\|x - y\| \geqq \epsilon$, then there exists a point, w, in $[x, y]^*$ such that $\|w\| < 1$.*

PROOF. Suppose (1) is true, and suppose that $\epsilon > 0$. Let $c = \frac{1}{2}\epsilon$. Then there exists a number, δ, such that $0 < \delta \leqq 1$, and if $\|r\| = \|s\| = 1$ and $\|r - s\| \geqq c$ then there exists a point t in $[r, s]^*$ such that $\|t\| \leqq 1 - \delta$.

Now, suppose that x and y are points such that $\|x\| = \|y\| = 1 + \delta$ and $\|x - y\| \geqq \epsilon$. Let

$$r = \frac{x}{1 + \delta} \quad \text{and} \quad s = \frac{y}{1 + \delta} \cdot$$

Then $\|r\| = \|s\| = 1$ and $\|r - s\| \geqq c$. Let t be a point of $[r, s]^*$ such that $\|t\| \leqq 1 - \delta$, and let $w = (1 + \delta)t$. Then w is in $[x, y]^*$ and $\|w\| \leqq (1 + \delta)(1 - \delta) < 1$. Thus (1) implies (2).

Now, suppose (2) is true, and suppose $\epsilon > 0$. Then there exists a number $\delta > 0$ such that if $\|x\| = \|y\| = 1 + \delta$ and $\|x - y\| \geqq \epsilon$, then there exists a point w in $[x, y]^*$ such that $\|w\| < 1$.

Suppose r and s are points such that $\|r\| = \|s\| = 1$ and $\|r - s\| \geqq \epsilon$, and let:

$$d = 1 - \frac{1}{1+\delta}; \qquad x = (1+\delta)r; \qquad y = (1+\delta)s.$$

Then $\|x-y\| \geqq \epsilon$, and there is a point w in $[x, y]^*$ such that $\|w\| < 1$. Now, if $t = w/(1+\delta)$, then t is in $[r, s]^*$ and $\|t\| \leqq 1-d$.

Thus (2) implies (1).

THEOREM 3.3. *Suppose that S is weakly uniformly convex, and suppose that M is a closed, norm convex point set at a distance* 1 *from N. Then M contains only one point Q such that $\|Q\| = 1$.*

PROOF. Since S is weakly uniformly convex, M does not contain two points of unit length.

Suppose M contains no point of unit length, and suppose that $\{P_i\}$ is a sequence of points of M such that $\{\|P_i\|\}$ converges to 1. Then $\{P_i\}$ is not a Cauchy sequence, and there exists a number $r > 0$ such that if J is a positive integer then there exist positive integers i and $j \geqq J$ such that $\|P_i - P_j\| \geqq r$.

By Theorem 3.2, there exists a number $h > 0$ such that if $\|u\| = \|v\| = 1+h$ and $\|u-v\| \geqq r$, then there exists a point w in $[u, v]^*$ such that $\|w\| \leqq 1$.

Let P and R be points of $\{P_i\}$ such that:

$$\begin{aligned} \|P\| &\geqq \|R\|, \\ \|P - R\| &\geqq r, \\ \|P\| &< 1 + h, \\ \|P\| &< 1 + \frac{rh}{2(1+h)} \cdot \end{aligned}$$

Let T be a point of $[N, P]$ such that $\|P-T\| = \|T-R\| = s$. Then $r/2 \leqq s \leqq \|P\|$.

Now, if:

$$P' = \frac{1+h}{s}(P-T),$$

$$R' = \frac{1+h}{s}(R-T),$$

then $\|P'\| = \|R'\| = 1+h$, and $\|P'-R'\| \geqq r$. Hence there exists a point, K', of $[P', R']^*$ such that $\|K'\| < 1$.

Let $K = T + (s/(1+h)) \cdot K'$. Then it follows that K is in $[P, R]^*$, and hence K is in M.

Now,

$$
\begin{aligned}
\|K - T\| + \frac{sh}{1+h} &\leqq \frac{s}{1+h}\|K'\| + \frac{sh}{1+h} \\
&\leqq s \\
&= \|P - T\|.
\end{aligned}
$$

Hence

$$\|K - T\| \leqq \|P - T\| - \frac{sh}{1+h}\cdot$$

Also,

$$
\begin{aligned}
\|K\| &\leqq \|T\| + \|P - T\| - \frac{sh}{1+h} \\
&< 1 + \frac{rh}{2(1+h)} - \frac{sh}{1+h} \\
&= 1 - \left(s - \frac{r}{2}\right)\cdot\frac{h}{1+h} \\
&\leqq 1.
\end{aligned}
$$

Therefore $\|K\| < 1$, which is a contradiction since K is in M.

Thus $\{P_i\}$ is a Cauchy sequence. Since $\{\|P_i\|\}$ converges to 1, the sequential limit, Q, of $\{P_i\}$, has norm 1, and is the point of M nearest to N.

We note that if S is not reflexive, then S contains a closed convex point set M at a distance 1 from N such that if Q is in M then there is a point P of M such that $\|P\| < \|Q\|$. Since l_1 is not reflexive but is weakly uniformly convex, the condition of Theorem 3.3 that M be norm convex cannot be replaced by the condition that M be convex.

Reference

1. J. A. Clarkson, *Uniformly convex spaces*, Trans. Amer. Math. Soc. **40** (1936), 396–414.

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