PERMANENTS OF DIRECT PRODUCTS[1]

MARVIN MARCUS

1. **Results.** It is well known [2] that if A and B are n and m-square matrices respectively then

$$\det(A \otimes B) = (\det(A))^m(\det(B))^n \tag{1}$$

where $A \otimes B$ is the tensor or direct product of A and B. By taking absolute values on both sides of (1) we can rewrite the equality as

$$|\det(A \otimes B)|^2 = (\det(AA^*))^m(\det(B^*B))^n, \tag{2}$$

where A^* is the conjugate transpose of A.

The main result is a direct extension of (2) to permanents. In general, equality will not be maintained, and the cases of equality will require a somewhat delicate analysis.

THEOREM 1. *If A and B are n-square and m-square complex matrices respectively then*

$$|\operatorname{per}(A \otimes B)|^2 \leqq (\operatorname{per}(AA^*))^m(\operatorname{per}(B^*B))^n. \tag{3}$$

Equality holds in (3) *if and only if either*

(a) *A has a zero row or B has a zero column, or*

(b) *A and B are both generalized permutation matrices, i.e., each of A and B is a product of a diagonal matrix and a permutation matrix.*

The inequality (1) should also be compared to a recent abstract [1] in which the following result is announced:

$$\operatorname{per}(A \otimes B) \geqq (\operatorname{per}(A))^m(\operatorname{per}(B))^n \tag{4}$$

where A and B are assumed to have non-negative entries.

A lower bound of the type (4) is also available for positive semi-definite hermitian matrices.

THEOREM 2. *If A and B are positive semi-definite hermitian n-square and m-square matrices respectively then*

$$\operatorname{per}(A \otimes B) \geqq \left(\frac{1}{n!}\right)^m \left(\frac{1}{m!}\right)^n (\operatorname{per}(A))^m(\operatorname{per}(B))^n. \tag{5}$$

Equality holds in (5) *if and only if at least one of A and B has a zero row.*

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In §3 we give a combinatorial application of Theorem 1.

2. **Proofs.** Let $e_1, \cdots, e_n$ be the unit n-tuples, $e_i=(\delta_{1i}, \cdots, \delta_{ni})$, and let $\epsilon_i=(\delta_{i1}, \cdots, \delta_{im})$, $i=1, \cdots, m$, be the unit m-tuples. In general we will lexicographically index the rows and columns of $A \otimes B$ by the set Γ whose elements are all the sequences $\alpha=(\alpha_1, \alpha_2)$, $1 \leqq \alpha_1 \leqq n$, $1 \leqq \alpha_2 \leqq m$. Row α of $A \otimes B$ is $A_{(\alpha_1)} \otimes B_{(\alpha_2)}$ where $A_{(i)}$ is the ith row of A. Similarly column α of $A \otimes B$ is $A^{(\alpha_1)} \otimes B^{(\alpha_2)}$ where $A^{(i)}$ is the ith column of A. To prove Theorem 1 we use a result in [3] that states that

$$|\operatorname{per}(XY)|^2 \leqq \operatorname{per}(XX^*) \operatorname{per}(Y^*Y) \tag{6}$$

for any two matrices X and Y. Equality can hold in (6) only if a row of X or a column of Y is zero, or X^* can be obtained from Y by post-multiplication with a generalized permutation matrix. Then directly applying (6) to $A \otimes I_m$ and $I_n \otimes B$ we have

$$\begin{aligned} |\operatorname{per}(A \otimes B)|^2 &= |\operatorname{per}((A \otimes I_m)(I_n \otimes B))|^2 \\ &\leqq \operatorname{per}((A \otimes I_m)(A \otimes I_m)^*) \operatorname{per}((I_n \otimes B)^*(I_n \otimes B)) \\ &= \operatorname{per}(AA^* \otimes I_m)\operatorname{per}(I_n \otimes B^*B). \end{aligned} \tag{7}$$

It is obvious from the structure of $I_n \otimes B^*B$ that per $(I_n \otimes B^*B) = (\text{per } (B^*B))^n$. On the other hand $X \otimes Y$ is always permutation equivalent to $Y \otimes X$, and since the permanent is unaltered by permutations it follows that per $(AA^* \otimes I_m) = (\text{per } (AA^*))^m$. The inequality (3) then follows directly from (7). To settle the cases of equality in (3) we use the result quoted for the cases of equality in (6). Thus equality holds in (3) only if

(a) a row of $A \otimes I_m$ or a column of $I_n \otimes B$ is zero, or

(b) the following equality holds

$$A^* \otimes I_m = (I_n \otimes B)\, DP \tag{8}$$

where D and P are mn-square diagonal and permutation matrices respectively. According to our previous remarks if row α of $A \otimes I_m$ is zero then $A_{(\alpha_1)} \otimes \epsilon_{\alpha_2}=0$. But this obviously implies that $A_{(\alpha_1)}=0$, i.e., that a row of A is zero. Similarly, if a column of $I_n \otimes B$ is zero it follows that a column of B must be zero. Thus let us assume that no row of A and no column of B is zero. Then the equality in (3) implies that (8) holds. But (8) is precisely the same as saying that for an appropriate permutation σ of Γ and suitable constants d_α, $\alpha \in \Gamma$,

$$(A^* \otimes I_m)^{(\alpha)} = d_{\sigma(\alpha)}(I_n \otimes B)^{\sigma(\alpha)}, \qquad \alpha \in \Gamma;$$

that is,

$$A^{*(\alpha_1)} \otimes \epsilon_{\alpha_2} = d_{\sigma(\alpha)} e_{\sigma(\alpha)_1} \otimes B^{(\sigma(\alpha)_2)}, \qquad \alpha \in \Gamma, \tag{9}$$

where if $\beta = (\beta_1, \beta_2) = \sigma(\alpha)$ then $\sigma(\alpha)_i = \beta_i$, $i = 1, 2$. The equality (9) can be restated

$$\overline{A}_{(\alpha_1)} \otimes \epsilon_{\alpha_2} = d_{\sigma(\alpha)} e_{\sigma(\alpha)_1} \otimes B^{\sigma(\alpha)_2}, \quad \text{for all } \alpha \in \Gamma, \tag{10}$$

where the bar in the first term indicates the complex conjugate. Now no d_α can be 0, $\alpha \in \Gamma$, otherwise (10) would imply that A has a zero row, our previous case. Moreover, since σ is a permutation of Γ, $\sigma(\alpha)_2$ varies over $1, \cdots, m$ as α varies through Γ. It follows from (10) that

$$A_{(t)} = a_t e_{i_t}, \qquad t = 1, \cdots, n,$$
$$B^{(t)} = b_t \epsilon_{j_t}, \qquad t = 1, \cdots, m,$$

for appropriate sequences $(i_1, \cdots, i_n)$, $1 \leqq i_t \leqq n$, and $(j_1, \cdots, j_m)$, $1 \leqq j_t \leqq m$, and nonzero constants a_t and b_t. It follows that the (α, β) entry of $A \otimes B$, α, β in Γ, is

$$\begin{aligned} (A_{(\alpha_1)} \otimes \epsilon_{\alpha_2}, e_{\beta_1} \otimes B^{(\beta_2)}) &= a_{\alpha_1} d_{\beta_2} (e_{i_{\alpha_1}}, e_{\beta_1})(\epsilon_{\alpha_2}, \epsilon_{j_{\beta_2}}) \\ &= a_{\alpha_1} d_{\beta_2} \delta_{i_{\alpha_1} \beta_1} \delta_{\alpha_2 j_{\beta_2}} \end{aligned} \tag{11}$$

where we have used the standard inner products for the various sequence spaces. Suppose first that $(i_1, \cdots, i_n)$ omits an integer q, $1 \leqq q \leqq n$. Then the (α_1, α_2), (q, β_2) entry of $A \otimes B$ is

$$a_{\alpha_1} d_{\beta_2} \delta_{i_{\alpha_1} q} \delta_{\alpha_2 j_{\beta_2}} = 0,$$

according to (11). That is, column (q, β_2) of $A \otimes B$ is zero. But then per $(A \otimes B) = 0$ and it follows that at least one of per (AA^*) or per $(B^*B) = 0$. (Recall that we are assuming equality in (3)).

But according to a recent inequality [**4**],

$$\operatorname{per}(AA^*) \geqq \prod_{i=1}^{n} (A_{(i)}, A_{(i)})$$

and

$$\operatorname{per}(B^*B) \geqq \prod_{i=1}^{m} (B^{(i)}, B^{(i)}).$$

It follows that A must have a zero row or B a zero column, again the previous case. Thus $(i_1, \cdots, i_n)$ can omit no integer q, $1 \leqq q \leqq n$, and thus must be $1, \cdots, n$ in some order. Similarly, if $(j_1, \cdots, j_m)$ were to omit p, $1 \leqq p \leqq m$, it would follow from (11) that the (α_1, p), (β_1, β_2) entry of $A \otimes B$ is

$$a_{\alpha_1} d_{\beta_2} \delta_{i_{\alpha_1}\beta_1} \delta_{p j_{\beta_2}} = 0.$$

Thus row (α_1, p) of $A \otimes B$ would be zero and once again we could conclude that A would have to have a zero row or B a zero column. Thus $(j_1, \cdots, j_m)$ must be a permutation of $1, \cdots, m$. In other words, both A and B must be generalized permutation matrices.

Suppose, conversely that A and B are generalized permutation matrices,

$$A = QD, \qquad B = RK$$

where $D = \text{diag}\,(a_1, \cdots, a_n)$, $K = \text{diag}\,(b_1, \cdots, b_m)$ and Q and R are n-square and m-square permutation matrices respectively.

Then

$$\begin{aligned} \text{per}(A \otimes B) &= \text{per}(QD \otimes RK) \\ &= \text{per}((Q \otimes R)(D \otimes K)) \\ &= \text{per}(D \otimes K) \\ &= \prod_{i=1}^{n} \prod_{j=1}^{m} a_i b_j. \end{aligned}$$

On the other hand,

$$\begin{aligned} \text{per}(AA^*) &= \text{per}(QDD^*Q^*) \\ &= \text{per}(DD^*) \\ &= \prod_{i=1}^{n} |a_i|^2, \\ \text{per}(B^*B) &= \prod_{i=1}^{m} |b_i|^2, \end{aligned}$$

and the equality in (3) holds. If either A has a zero row or B a zero column then both sides of (3) are 0. This completes the proof of Theorem 1.

To prove Theorem 2 we observe first that $A \otimes B$ is also positive semi-definite hermitian. It is proved in [**4**] that for any positive semi-definite hermitian matrix A

$$\prod_{i=1}^{n} a_{ii} \leqq \text{per}(A) \leqq n! \prod_{i=1}^{n} a_{ii}. \tag{12}$$

The lower inequality holds in (12) if and only if A is a diagonal matrix or A has a zero row. The upper inequality holds if and only if A has a zero row or A is of rank 1. Applying the inequalities (12) to $A \otimes B$ we have

$$
(13)\qquad
\begin{aligned}
\operatorname{per}(A \otimes B) &\geqq \prod_{\alpha\in\Gamma} (A \otimes B)_{\alpha\alpha} \\
&= \prod_{\alpha_1=1}^{n} \prod_{\alpha_2=1}^{m} a_{\alpha_1\alpha_1} b_{\alpha_2\alpha_2} \\
&= \left(\prod_{i=1}^{n} a_{ii}\right)^{m} \left(\prod_{i=1}^{m} b_{ii}\right)^{n} \\
&\geqq \left(\frac{\operatorname{per}(A)}{n!}\right)^{m} \left(\frac{\operatorname{per}(B)}{m!}\right)^{n},
\end{aligned}
$$

the required inequality. We will have equality throughout (13) if $A \otimes B$ has a zero row. If $A \otimes B$ has no zero row then equality throughout (13) would require that $A \otimes B$ be a diagonal matrix of rank 1, an obvious impossibility. On the other hand, $A \otimes B$ can have a zero row if and only if either A or B does. This completes the proof of Theorem 2.

3. **A combinatorial application.** Let $S = \{a_1, \cdots, a_n\}$ and let $S_1, \cdots, S_n$ be subsets of S. Similarly let $T = \{b_1, \cdots, b_m\}$ and let $T_1, \cdots, T_m$ be subsets of T. The incidence matrix for the configuration S is defined to be the n-square 0-1 matrix A whose (i, j) entry is 1 or 0 according as $a_i \in S_j$ or $a_i \notin S_j$. We can similarly define the m-square incidence matrix B for the configuration T. Consider the cartesian product set $S \times T$ and the nm subsets $S_i \times T_j$, $i = 1, \cdots, n$, $j = 1, \cdots, m$. The incidence matrix for this configuration is constructed as follows. If α and β are in Γ then $(a_{\alpha_1}, b_{\alpha_2}) \in S_{\beta_1} \times T_{\beta_2}$ if and only if $a_{\alpha_1} \in S_{\beta_1}$ and $b_{\alpha_2} \in T_{\beta_2}$. In other words, the α, β entry of the incidence matrix for the cartesian product configuration is $\delta_{\alpha_1\beta_1}\delta_{\alpha_2\beta_2}$. But this is just the (α, β) entry of $A \otimes B$. Thus $A \otimes B$ is the incidence matrix for the $S \times T$ configuration.

A system of distinct representatives (SDR) for the subsets $S_1, \cdots, S_n$ [5] is an ordered selection

$$
a_{\phi(1)}, \cdots, a_{\phi(n)}, \ a_{\phi(i)} \in S_i, \qquad i = 1, \cdots, n.
$$

It is an immediate consequence of the definition that the number of SDR's for the subsets $S_1, \cdots, S_n$ is just per (A) [5, p. 54].

It is clear that Theorem 1 implies the following result.

THEOREM 3. *Let p denote the number of SDR's for the cartesian product configuration, $S \times T$, and let A and B be the incidence matrices for the S and T configurations respectively. Then*

$$
(14)\qquad p \leqq (\operatorname{per}(AA^*))^{m/2} \, (\operatorname{per}(B^*B))^{n/2}.
$$

Equality can hold in (14) *if and only if*

(a) *the subsets* $S_1, \cdots, S_n$ *all omit some* a_i, *or*

(b) *some* T_j *is empty, or*

(c) $S_i = \{a_{\phi(i)}\}$, $i = 1, \cdots, n$, *and* $T_j = \{b_{\theta(j)}\}$, $j = 1, \cdots, m$, *where* ϕ *and* θ *are permutations of* $1, \cdots, n$ *and* $1, \cdots, m$ *respectively.*

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University of California, Santa Barbara