PARTIALLY ORDERED GROUPS OF THE SECOND AND THIRD KINDS[1]

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1. **Introduction.** Let G be both a group and a partially ordered set. An element a of G is called a *left* [*right*] *conserver* if

$$x \leqq y(x, y \in G) \Rightarrow ax \leqq ay[xa \leqq ya]$$

and a *left* [*right*] *inverter* if

$$x \leqq y(x, y \in G) \Rightarrow ax \geqq ay[xa \geqq ya].$$

We shall call an element of G a *conserver* [*inverter*] if it is both a left and a right conserver [inverter].

If every element of G is a conserver, then G is a partially ordered group (abbreviated "po-group") in the usual sense; we shall also say that G is a *po-group of the first kind.* If every element of G is a conserver or an inverter, and not every element of G is a conserver, then we shall call G a *po-group of the second kind.* A familiar example is the multiplicative group of all nonzero real numbers with the usual ordering. The stipulation that not every element of G is a conserver excludes the possibility that G be trivially ordered, and it is then clear that no element of G can be both a conserver and an inverter.

The structure of totally ordered groups ("o-groups") of the second kind has been reduced to that of o-groups of the first kind by J. A. H. Shepperd [**1**]. (What he calls a "betweenness group" is either an o-group of the first or second kinds, or a finite group of order 4.) The first main result of the present note (Theorem 1) is an extension of Shepperd's result from o-groups to po-groups. The proof has also been simplified by avoiding reference to the betweenness relation.

Totally ordered semigroups ("o-semigroups") of the second kind have been considered by the author [**2**] in the commutative case, and by J. Gilder [**3**] and K. Keimel [**4**] in general. Following Gilder's terminology, we define a *po-group of the third kind* to be a group G endowed with a nontrivial partial order, such that each element of G is either a left conserver or a left inverter, and also either a right conserver or a right inverter, and such that G contains an element which conserves on one side and inverts on the other. Theorem 2 gives a reduction of these to po-groups of the first kind.

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Added in proof. The author regrets that he was not aware at the time of writing this paper that a result equivalent to Theorem 1 below had been obtained by J. F. Andrus and A. T. Butson [**5**] for a connected po-group of the second kind. The two approaches are also different, however, that the equivalence of the results is not apparent. Their subgroup S_0 is the (directed) po-subgroup of my subgroup H generated by its positive cone H_+. My subset I_- of $G\backslash H$ is the union of those cosets of S_0 in G which belong to their subset T_0 of the factor group G/S_0. Removing their requirement of connectedness actually simplifies more than it complicates. Thus the six properties (i)–(vi) which T_0 must have in their Theorem 5 reduce to (i), (iv), (v), and the requirement that there exists a subgroup H of index 2 in P $(=G/S_0)$ such that $T_0 \leqq G\backslash H$. (Incidently, (v) should read "$a+T_0 = T_a$".) This is an immediate consequence of Theorem 1 below, in the case $H_+=0$.

2. **Partially ordered groups of the second kind.** We denote the identity element of G by e, and set

$$G_+ = \{a \in G: a \geqq e\}, \qquad G_- = \{a \in G: a \leqq e\}.$$

For any subset A of G, we let $A_+=A\cap G_+$, $A_-=A\cap G_-$, and $A^{-1}=\{a^{-1}:a\in A\}$. A (possibly empty) subset A of G is called an *upper* [*lower*] *class in* G if $a\in A$, $x\in G$, and $a<x\,[a>x]$ imply $x\in A$. The empty set is denoted by $\varnothing$, and $A\backslash B$ means the set of elements of A not in B.

By the *order dual* G^* of G we mean the group G endowed with the dual $\leqq^*$ of $\leqq (a\leqq^* b \Leftrightarrow b\leqq a)$. G and G^* have the same sets of left [right] conservers and inverters.

THEOREM 1. *Let G be a po-group of the second kind. Let $H[I]$ be the set of conservers [inverters] of G. Then H is a subgroup of G of index* 2, *and I is its other coset. H is a po-group of the first kind, and H_+ is normal in G. H and I are convex subsets of G, and, by passing to the order dual of G if necessary, we can assume that H is an upper class and I a lower class in G. In particular, $H_+=G_+$, $I_+=\varnothing$, and I_- is a lower class in G. The set I_- has the following properties:*

(N1) *I_- is normal in G.*

(N2) $I_-^{-1}=I_-$.

(N3) *I_- contains H_+I_-, I_-H_+, H_-I_-, and I_-H_-.*

The order relation $\leqq$ can be described in terms of H_+ and I_- as follows:

(O1) *If $x\in H$, $y\in H$, then $x\leqq y \Leftrightarrow x^{-1}y$ (or yx^{-1}) $\in H_+$.*

(O2) *If $x\in I$, $y\in I$, then $x\leqq y\Leftrightarrow xy^{-1}$ (or $y^{-1}x$) $\in H_+$.*

(O3) *If $x\in I$, $y\in H$, then $x\leqq y\Leftrightarrow xy^{-1}$ (or $y^{-1}x$ or $x^{-1}y$ or yx^{-1}) $\in I_-$.*

(O4) *If* $x\in H$, $y\in I$, *then* $x\leqq y$ *never holds.*
If H is directed, then I_- *must be I or* $\emptyset$. *If* $I_-=I$, *then every element of I is less than every element of H. If* $I_-=\emptyset$, *then no element of I is comparable with any element of H.*

Conversely, let G be a group containing a subgroup H of index 2, *and let* $I=G\backslash H$. *Assume that H is a po-group of the first kind, and that its positive cone* H_+ *is normal in G. Let* I_- *be a subset of I having properties* (N1–3). *Define* $\leqq$ *in G by* (O1–4). *This agrees with the given partial order in H by* (O1), *and G becomes thereby a po-group of the second kind such that H*[*I*] *is the set of conservers* [*inverters*] *of G.*

REMARKS. (1) If (N1) and (N2) hold, and if I_- contains any one of the four product sets in (N3), then it contains the other three.

(2) Regarding the parenthetical assertions in (O3), we note that if I_- is any subset of I satisfying (N1) and (N2), and if any one of the four products xy^{-1}, $y^{-1}x$, $x^{-1}y$, yx^{-1} belong to I_-, then so do the remaining three. A similar remark applies to (O1) and (O2), since H_+ is normal in G.

PROOF. Evidently the product of two conservers or of two inverters is a conserver, while that of a conserver and an inverter is an inverter. Since the identity element e of G is a conserver, the inverse of a conserver [inverter] must be of the same type. From these remarks it is clear that H is a subgroup of G of index 2, that $I=G\backslash H$, and that H is a po-group of the first kind.

If $p\in H_+$ and $u\in I$, then from $e\leqq p$ we have $u\geqq pu$ and $e=u^{-1}u\leqq u^{-1}pu$. Thus $u^{-1}H_+u\subseteq H_+$. Since H_+ is normal in H, this shows that it is normal in G.

To show that H is convex in G, it clearly suffices to show that $e<u<h(h\in H,\ u\in I)$ is impossible. Multiplying $e<u<h$ on the left by the inverter u, and on the right by the conserver h, we obtain $u>u^2>uh$ and $h<uh<h^2$. But this yields $u>uh>h$, contrary to $u<h$.

To show that I is convex in G, suppose that $u>h>u'(h\in H;\ u,\ u'\in I)$. Then $e<hu^{-1}<u'u^{-1}$. Since $u'u^{-1}\in H$ and $hu^{-1}\in I$, this contradicts the convexity of H.

From $G=H\cup I$ it follows that H must be either an upper class or a lower class in G. By passing to the order dual of G, if necessary, we can assume that H is an upper class. Then I is a lower class. Since $e\in H$, we have $G_+\subseteq H$, and hence $H_+=G_+$ and $I_+=\emptyset$. I_- is clearly a lower class in I, hence also in G.

If $u\in I_-$ and $v\in I$, then from $u<e$ we have $uv>v$ and $v^{-1}uv<v^{-1}v=e$, hence $v^{-1}uv\in I_-$. Similarly, $h^{-1}uh\in I_-$ for every h in H, which proves (N1). To show (N2), we note that $u<e$ $(u\in I)$ implies $e=uu^{-1}>eu^{-1}=u^{-1}$, hence $u^{-1}\in I_-$. By Remark (1), to establish (N3) we need only show that $H_+I_- \subseteq I_-$. From $h > e$, $u < e$ ($h \in H$, $u \in I$), we have $hu < eu = u < e$, so that $hu \in I_-$.

(O1) is a standard fact about po-groups, and (O4) is just the assertion that H is an upper class in G. To show (O2), we note that $x \leqq y$ is equivalent to $xy^{-1} \geqq e$, since $y \in I$. To show (O3), we observe that $x \leqq y$ is now equivalent to $xy^{-1} \leqq yy^{-1} = e$, since $y \in H$.

If H is directed, then $H = H_+H_-$, and (N3) implies that $HI_- \subseteq I_-$. If $I_- \neq \emptyset$, let $u \in I_-$. Then $I = Hu \subseteq I_-$, whence $I_- = I$.

Turning to the converse, let H be a subgroup of G of index 2, and let H be a po-group of the first kind such that H_+ is normal in G. Let $I = G \backslash H$. Let I_- be a subset of I having properties (N1–3), and define $\leqq$ in G by (O1–4). (O1) asserts that the restriction of $\leqq$ to H shall coincide with the given partial ordering of H.

That the relation $\leqq$ is reflexive and antisymmetric is clear. To prove that it is transitive, let $x \leqq y$ and $y \leqq z$ ($x, y, z \in G$). (O4) implies that if $x \in H$ then $y \in H$, and if $y \in H$ then $z \in H$. Since we do not need to consider the case $x, y, z \in H$, we are left with three cases.

Case $x \in I$, $y \in H$, $z \in H$. By (O3) and (O1) we have $x^{-1}y \in I_-$ and $y^{-1}z \in H_+$. By (N3), $x^{-1}z = (x^{-1}y)(y^{-1}z) \in I_-$, and $x \leqq z$ by (O3).

Case $x \in I$, $y \in I$, $z \in H$. By (O2) and (O3) we have $xy^{-1} \in H_+$ and $yz^{-1} \in I_-$. By (N3), $xz^{-1} = (xy^{-1})(yz^{-1}) \in I_-$, and $x \leqq z$ by (O3).

Case $x \in I$, $y \in I$, $z \in I$. By (O2) we have $xy^{-1} \in H_+$ and $yz^{-1} \in H_+$. Hence $xz^{-1} = (xy^{-1})(yz^{-1}) \in H_+$, and $x \leqq z$ by (O2).

Hence G is a po-set under $\leqq$. All that remains is to show that every element of $H[I]$ is a conserver [inverter]. Since every element of H is the product of two elements of I, it suffices to show that every element of I is an inverter.

Let $u \in I$, and let $x \leqq y$. The case $x \in H$, $y \in I$ is excluded by (O4), and we consider the remaining three.

Case $x \in H$, $y \in H$. By (O1), $x^{-1}y$ and $yx^{-1} \in H_+$. Hence $(ux)^{-1}(uy)$ and $(yu)(xu)^{-1} \in H_+$, and we infer from (O2) that $uy \leqq ux$ and $yu \leqq xu$.

Case $x \in I$, $y \in I$. By (O2), xy^{-1} and $y^{-1}x \in H_+$. Hence $(xu)(yu)^{-1}$ and $(uy)^{-1}(ux) \in H_+$, and we infer from (O1) that $yu \leqq xu$ and $uy \leqq ux$.

Case $x \in I$, $y \in H$. By (O3), $x^{-1}y$ and $yx^{-1} \in I_-$. Hence $(ux)^{-1}(uy)$ and $(yu)(xu)^{-1} \in I_-$, and we infer from (O3) that $uy \leqq ux$ and $yu \leqq xu$.

This concludes the proof of the theorem.

Let us consider all possible ways of extending a given po-group H of the first kind to a po-group G of the second kind, such that H is the set of conservers of G. In the first place, G must be an extension of H by the cyclic group C_2 of order 2; the Schreier theory tells us how to find all such. Call G "suitable" if H_+ is normal in G; there is at least one suitable G, namely the direct product $H\times C_2$. Any suitable G can be partially ordered in the desired fashion by choosing I_- so as to satisfy (N1–3). This can always be done by choosing I or $\varnothing$ for I_-, and these are the only possibilities if H is directed. If G itself is to be directed, only $I_-=I$ is possible, and then every element of H exceeds every element of I. In this case we note that G will be lattice-ordered or totally ordered if and only if the same holds for H.

If H is trivially ordered, then $I_-\neq\varnothing$, since G cannot be trivially ordered. As a simple example with $I_-\neq I$, let G be the infinite cyclic group generated by a, let H be the subgroup generated by a^2, and let $I_-=\{a, a^{-1}\}$. The resulting partial order on G has a saw-tooth nature:

$$\cdots > a^{-3} < a^{-2} > a^{-1} < e > a < a^2 > a^3 < \cdots .$$

3. **Partially ordered groups of the third kind.** We define the following four subsets of a po-group G of the third kind. Let $C_2=\{0, 1\}$ be the additive group of integers mod 2, so that $1+1=0$. For i and j in C_2 let G_{ij} be the set of all elements a of G such that a is a left conserver if $i=0$, a left inverter if $i=1$, a right conserver if $j=0$, and a right inverter if $j=1$.

From the way left and right conservers and inverters multiply,

$$G_{ij}G_{kl} = G_{i+k,j+l} \qquad (i, j, k, l \in C_2).$$

By definition of po-group of the third kind,

$$G = G_{00} \cup G_{01} \cup G_{10} \cup G_{11}, \qquad G_{01} \cup G_{10} \neq \varnothing.$$

If G_{01} and G_{10} are $\neq\varnothing$, then $G_{11}\neq\varnothing$.

THEOREM 2. *Let G be a po-group of the third kind, with G_{ij} as defined above. Then G_{00} is a normal subgroup of G, and is a po-group of the first kind. If $G_{11}=\varnothing$, then $G/G_{00}\cong C_2$. If $G_{11}\neq\varnothing$, then $G/G_{00}\cong C_2\times C_2$, and $G_{00}\cup G_{11}$ is a normal po-subgroup of G of the second kind. The positive cone P of G_{00} satisfies the following conditions:*

(N′1) *if $a\in G_{00}\cup G_{11}$, then $aPa^{-1}\subseteq P$;*

(N′2) *if $a\in G_{01}\cup G_{10}$, then $aPa^{-1}\subseteq P^{-1}$.*

No two elements belonging to different G_{ij} are comparable. Within each G_{ij}, the order relation is given in terms of P as follows:

(O′1) *if $x, y\in G_{00}$ or $x, y\in G_{01}$, then $x\leqq y\Leftrightarrow x^{-1}y\in P$;*

(O′2) *if $x, y\in G_{10}$ or $x, y\in G_{11}$, then $x\leqq y\Leftrightarrow x^{-1}y\in P^{-1}$.*

Conversely, let G be a group containing a normal subgroup H_{00} such that $G/H_{00}\cong C_2$ or $C_2\times C_2$. In the latter event, let H_{ij} be the coset of H_{00} in G mapped into the element (i, j) of $C_2\times C_2$. Assume that H_{00} is a *po-group of the first kind such that its positive part P satisfies* (N′1–2), *with G_{ij} replaced by H_{ij}. Define a relation $\leqq$ on G by* (O′1–2), *similarly modified, with $\leqq$ never holding between elements of distinct H_{ij}. Then G becomes a po-group of the third kind, with $G_{ij}=H_{ij}$. The same holds in the event $G/H_{00}\cong C_2$ if we let $H_{11}=\varnothing$, and either $H_{01}=\varnothing$ or $H_{10}=\varnothing$.*

PROOF. The first three sentences are obvious. (N′1) then follows from Theorem 1. To show (N′2), let $p\in P$ and let $a\in G_{01}$. From $e<p$ we have $a<ap$, since a is a left conserver, and hence $e>apa^{-1}$, since a is a right inverter. Thus $apa^{-1}\in P^{-1}$. The proof for a in G_{10} is similar.

We note that the identity element e of G cannot be comparable with any element of G_{01} or G_{10}. For if $e<a$ $(a\in G_{01})$, then $a<a^2$ since a is a left conserver, and $a>a^2$ since a is a right inverter. The argument is similar if $e>a$, or if $a\in G_{10}$. Moreover, e cannot be comparable with an element a of G_{11}. For suppose $a<e$. Let $b\in G_{01}$. Then $ba<b$ and $bab^{-1}>e$, since $b^{-1}\in G_{01}$. But $bab^{-1}\in G_{01}G_{11}G_{01}=G_{11}$, and $a<e<bab^{-1}$ would violate the convexity of $I=G_{11}$ in the po-group $G_{00}\cup G_{11}$ of the second kind (Theorem 1). The argument is similar if $e<a$.

Now let a and b be any two elements of G such that $a<b$. Then $aa^{-1}<ba^{-1}$ or $aa^{-1}>ba^{-1}$, depending on whether a^{-1} is a right conserver or a right inverter. In either case, ba^{-1} is comparable with e, and so belongs to G_{00}. Hence a and b belong to the same coset G_{ij}.

To show (O′1), $x\leqq y\Leftrightarrow e\leqq x^{-1}y\Leftrightarrow x^{-1}y\in P$, since x is a left conserver. As for (O′2), x is a left inverter, and so $x\leqq y\Leftrightarrow e\geqq x^{-1}y\Leftrightarrow x^{-1}y\in P^{-1}$.

Proceeding to the converse, let us introduce the notation $P_0=P$, $P_1=P^{-1}$, where $0, 1\in C_2$. Then, for any k in C_2, $P_k^{-1}=P_{k+1}$. The modified rules (N′1–2) and (O′1–2) can then be condensed into single formulae:

(N′) if $a\in H_{ij}$, then $aPa^{-1}\subseteq P_{i+j}$;

(O′) if $x, y\in H_{kl}$, then $x\leqq y\Leftrightarrow x^{-1}y\in P_k$.

It is evident that $\leqq$ defined by (O′) is reflexive and symmetric. If $x\leqq y$ and $y\leqq z$, then x, y, and z all belong to the same H_{kl}, and $x^{-1}z=(x^{-1}y)(y^{-1}z)\in P_kP_k\subseteq P_k$, whence $\leqq$ is transitive.

To show that $H_{ij}=G_{ij}$, let $a\in H_{ij}$ and let $x\leqq y$. Then x and y belong to the same H_{kl} and $x^{-1}y\in P_k$. From $(ax)^{-1}(ay)=x^{-1}y\in P_k$, and ax, $ay\in H_{i+k,j+l}$, (O′) gives $ax\leqq ay$ if $i=0$ and $ay\leqq ax$ if $i=1$. Since this is independent of k and l, we conclude that a is a left conserver if $i=0$ and a left inverter if $i=1$.

From $(xa)^{-1}(ya)=a^{-1}(x^{-1}y)a\in P_{k+(i+j)}=P_{(i+k)+j}$ by (N′), and xa, $ya\in H_{i+k,j+l}$, we conclude from (O′) that $xa\leqq ya$ if $j=0$ and $ya\leqq xa$ if $j=1$. Hence a is a right conserver if $j=0$ and a right inverter if $j=1$.

Let H be a po-group of the first kind. We saw at the conclusion of §2 that H can be extended in at least one way to a po-group G of the second kind. This is not so if G is to be of the third kind. In fact it is possible if and only if there exists an automorphism of H the square of which is inner, and which maps the positive cone P of H into P^{-1}. This is always possible if H is abelian, since $x \rightarrow x^{-1}$ is then an automorphism with these properties. But it is impossible if H is a group every automorphism of which is inner. For example, let H be the group of rational matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

with $a > 0$, and define $P(H)$ to be the set of all such matrices with $a \geqq 1$.

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