A SCHWARZ LEMMA FOR BOUNDED SYMMETRIC DOMAINS

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The purpose of this note is to generalize Pick's invariant formulation of the classical Schwarz lemma. For the four classical types of bounded symmetric domains such a generalization was given by K. H. Look [3]; the present treatment will be independent of classification theory and will also include the exceptional Cartan domains. The results are also independent of the particular realization of the domain in C^n, they depend only on its structure as a hermitian manifold. The results will therefore be formulated for hermitian symmetric spaces of noncompact type; these are known to be in one-to-one correspondence with the holomorphic equivalence classes of bounded symmetric domains. We shall make use of Harish-Chandra's canonical realization of the hermitian symmetric spaces as bounded domains; this could perhaps be avoided, but it makes the proofs considerably simpler.

In the following $M=G/K$ will be a hermitian symmetric space of noncompact type; the identity coset will be denoted by p_0, $\mathfrak{g}$ and $\mathfrak{k}$ will denote the Lie algebras of G and K, respectively, and H_α, E_α, $\cdots$ will be a Weyl basis of $\mathfrak{g}$ with respect to a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{k}$. By a result of Harish-Chandra there exists a set Δ of strongly orthogonal roots of $\mathfrak{g}$ such that $\mathfrak{a}=\sum_{\alpha\in\Delta} R(E_\alpha+E_{-\alpha})$ is a Cartan subalgebra of the symmetric pair $(\mathfrak{g}, \mathfrak{k})$. So every point $p\in M$ can be represented in the form $p=k\exp(\sum_{\alpha\in\Delta} t_\alpha(E_\alpha+E_{-\alpha}))\cdot p_0$ with $k\in K$, $t_\alpha\geqq 0$.

For any p, $q\in M$ we denote by $d(p, q)$ the distance of p and q in the metric induced by the hermitian structure of M. In any realization of M as a complex domain this is the Bergman metric. We denote by $d^*(p, q)$ the Carathéodory distance, which is defined by

$$d^*(p, q) = \sup_{f\in F} d_U(f(p), f(q)),$$

where F is the family of all holomorphic maps of M into the unit disc $U\subset C$, and d_U is the Poincaré-Bergman distance function on U.

LEMMA. *Let* $p=\exp(\sum_{\alpha\in\Delta} t_\alpha(E_\alpha+E_{-\alpha}))\cdot p_0$, $t_\alpha\geqq 0$ $(\alpha\in\Delta)$. *Then*

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$$d(p_0, p) = \Big(\sum_{\alpha\in\Delta} t_\alpha^2\Big)^{1/2},$$
$$d^*(p_0, p) = \operatorname*{Max}_{\alpha\in\Delta} t_\alpha.$$

PROOF. The first statement follows from the known fact that the orbits of p_0 under one-parameter groups generated by elements of $\mathfrak{a}$ are geodesics; thus

$$\exp\left(s\Big(\sum_{\alpha\in\Delta} t_\alpha^2\Big)^{-1/2}\sum_{\alpha\in\Delta} t_\alpha(E_\alpha + E_{-\alpha})\right)\cdot p_0\left(0 \leqq s \leqq \Big(\sum_{\alpha\in\Delta} t_\alpha^2\Big)^{1/2}\right)$$

is a geodesic segment in arc-length parameters connecting p_0 and p.

For the second statement we use the Harish-Chandra realization of M as a bounded domain. We denote by Φ the set of positive roots of $\mathfrak{g}$ which are not roots of $\mathfrak{k}$, and by $\mathfrak{p}^-$ the complex subspace of $\mathfrak{g}^C$ spanned by the vectors $E_{-\alpha}(\alpha\in\Phi)$. The Harish-Chandra realization $\eta\colon M\to\mathfrak{p}^-$ is given for any $p=k\exp(\sum_{\alpha\in\Delta} t_\alpha(E_\alpha+E_{-\alpha}))\cdot p_0$ by $\eta(p) = \operatorname{ad}(k)\sum_{\alpha\in\Delta} r_\alpha E_{-\alpha}$, where $r_\alpha=\tanh t_\alpha(\alpha\in\Delta)$. We denote the domain $\eta(M)\subset\mathfrak{p}^-$ by D.

Now let p be as in the statement of the Lemma, and let $\alpha_0\in\Delta$ be such that $t_{\alpha_0}=\operatorname{Max}_{\alpha\in\Delta} t_\alpha$. We write $r_{\alpha_0}=\tanh t_{\alpha_0}$. Let $f\colon M\to U$ be a holomorphic function such that $f(p_0)=0$. Defining the function $\phi\colon U\to U$ by $\phi(z)=f(\eta^{-1}(zr_{\alpha_0}^{-1}\eta(p)))$ we have, by the classical Schwarz lemma, $|\phi(z)|\leqq|z|$ for all $z\in U$. In particular, for $z=r_{\alpha_0}$, it follows that $|f(p)|\leqq r_{\alpha_0}$. In the definition of the Carathéodory distance it is sufficient to consider functions $f\in F$ such that $f(p_0)=0$, since U is homogeneous. Hence, from what we just proved it follows that $d^*(p_0, p)\leqq d_U(0, r_{\alpha_0})=t_{\alpha_0}$.

On the other hand, let $g\colon D\to U$ be defined by $g(\sum_{\alpha\in\Phi} z_\alpha E_{-\alpha})=z_{\alpha_0}$, and let $f_1=g\circ\eta$. Then $f_1\in F$, and $d^*(p_0, p)\geqq d_U(f_1(p_0), f_1(p)) = d_U(0, r_{\alpha_0})=t_{\alpha_0}$, finishing the proof of the Lemma.

PROPOSITION 1. *Let M be a hermitian symmetric space of rank l and let $f\colon M\to M$ be a holomorphic function. Then, for any p, $q\in M$,*

$$d(f(p), f(q)) \leqq l^{1/2} d(p, q).$$

The constant $l^{1/2}$ is the best possible.

PROOF. Given any pair of points p_1, $p_2\in M$ we can find an element g in G such that $gp_1=p_0$, $gp_2=\exp(\sum_{\alpha\in\Delta} t_\alpha(E_\alpha+E_{-\alpha}))\cdot p_0$. By the Lemma it follows that

$$d(f(p), f(q)) \leqq l^{1/2} d^*(f(p), f(q)) \leqq l^{1/2} d^*(p, q) \leqq l^{1/2} d(p, q)$$

proving the first statement.

To see that $l^{1/2}$ is best possible, let $p = \exp t(E_{\alpha_0} + E_{-\alpha_0}) \cdot p_0$ with some $t > 0$ and $\alpha_0 \in \Delta$. Define $g: D \to D$ by $g(\sum_{\alpha \in \Phi} z_\alpha E_{-\alpha}) = z_{\alpha_0} \sum_{\alpha \in \Delta} E_{-\alpha}$, and let $f = \eta^{-1} \circ g \circ \eta$. By Lemma 1 we have $d(p_0, p) = t$ and $d(f(p_0), f(p)) = l^{1/2} t$, finishing the proof.

REMARK. The Proposition remains true, by the same proof, for holomorphic functions $f: M_1 \to M_2$ where M_1, M_2 are hermitian symmetric spaces and l is the rank of M_2.

Next we give an infinitesimal formulation of Proposition 1; here we are also able to prove an analogue of the "strong form" of the classical Schwarz lemma (cf. [**3**]). For every $p \in M$ we denote by M_p the space of real tangent vectors at p. M_p is a complex Euclidean space under the hermitian structure of M, we denote the length of a vector $X \in M_p$ by $\|X\|$.

PROPOSITION 2. *Let M be a hermitian symmetric space of rank l and let f: $M \to M$ be a holomorphic function. Then for all $p \in M$ and $X \in M_p$ we have $\|df(X)\| \leqq l^{1/2} \|X\|$, the constant $l^{1/2}$ being the best possible.*

If there exists a point $p \in M$ such that $\|df(X)\| \geqq \|X\|$ for all $X \in M_p$, then f is a holomorphic automorphism of M.

PROOF. The first statement follows from Proposition 1. To prove the second statement, let $g \in G$ be such that $gp = f(p)$. Then $h = g^{-1} \circ f$ maps M onto itself and keeps p fixed; since g^{-1} is an isometry, the hypothesis implies $\|dh(X)\| \geqq \|X\|$ for all $X \in M_p$, and hence $|\det(dh)_p| \geqq 1$. By a well-known theorem of H. Cartan and Carathéodory (e.g. [**1**, Chapter **1**]) it follows that h, and therefore also f, is a holomorphic automorphism of M, finishing the proof.

If $l = 1$, we have the following sharper version of the "strong form."

PROPOSITION 3. *Let M be a hermitian symmetric space of rank* 1 *and dimension n, and let f: $M \to M$ be a holomorphic function. If there exists a point $p \in M$ and n complex-linearly independent vectors $X_1, \cdots, X_n \in M_p$ such that $\|df(X_i)\| = \|X_i\|$ $(i = 1, \cdots, n)$, then f is a holomorphic automorphism of M.*

PROOF. Let $g \in G$ be such that $gp = f(p)$, and let $h = g^{-1} \circ f$. By Proposition 2, $(dh)_p$ is a linear contraction of the complex Euclidean space M_p. Denoting the adjoint transformation by $(dh)_p^*$ it follows that $A = I - (dh)_p (dh)_p^*$ is a positive semidefinite linear transformation on M_p. By our hypothesis $\|AX_i\| = 0$ $(i = 1, \cdots, n)$; this now implies $A = 0$. It follows that $(dh)_p$ is unitary, whence $|\det(dh)_p| = 1$, and by the above mentioned theorem of Cartan and Carathéodory the proof is finished.

References

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A CHARACTERIZATION OF TAME 2-SPHERES IN E^3

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In this note, the tame 2-spheres in E^3 are characterized partly in terms of homology and the arcs they contain. In a similar way, the compact 2-manifolds with boundary are characterized. If K is a finite topological 2-complex in E^3 and v is a vertex of K, then $\mathrm{St}\, v$ is the star of v, $\dot{\mathrm{S}}\mathrm{t}\, v$ is the open star of v, and $\mathrm{Lk}\, v = \mathrm{St}\, v - \dot{\mathrm{S}}\mathrm{t}\, v$ is the link of v. The trivial 1-dimensional homology group of K will be denoted by $H_1(K) = 0$.

An n-manifold with boundary is a separable metric space such that each point has a neighborhood whose closure is topologically equivalent to a closed n-cell.

Theorem 1. *Let K be a finite topological 2-complex in E^3 such that*
(i) *K is connected,*
(ii) *$\mathrm{Lk}\, v$ is connected for each vertex v in K,*
(iii) *$H_1(K) = 0$, and*
(iv) *K contains only tame arcs.*
Then K is either a disk or a 2-sphere.

Proof. Since K contains no wild arcs and $\mathrm{Lk}\, v$ is connected, each 1-simplex in K lies on exactly one or two 2-simplices in K [2]. Since

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