RETRACEABLE SETS AND RECURSIVE PERMUTATIONS

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1. By a *recursion property* of sets of natural numbers, we mean a property preserved under all recursive permutations of the natural numbers; while by a *weak-recursion property* we mean one which is preserved under recursive equivalence (see [**1**]). It is known [**1**] that *regressiveness* is a recursion property and even a weak-recursion property. We shall prove here that the property of being *retraceable* fails, by contrast, to be a recursion property, even for sets with recursively enumerable complement. (It is known from [**1**] that retraceability is not a *weak-recursion* property.) Our basic terminology and notation are as in [**1**].

We shall repeat here, for the reader's convenience, the definition of retraceability and regressiveness of sets of natural numbers. An infinite set α of natural numbers is *regressive* $\Leftrightarrow$ there is a nonrepetitive ordering $a_0, a_1, a_2, \cdots$ of the elements of α, and a partial recursive function p whose domain includes α, such that $p(a_0)=a_0$ and $(\forall n)\ (p(a_{n+1})=a_n)$. α is called *retraceable* provided it is regressive with respect to the ordering of its elements in increasing order of magnitude.

2. It is known [**1**, Proposition 10] that any recursively enumerable set with regressive complement is isomorphic to a recursively enumerable set whose complement is retraceable ("isomorphic to" meaning, of course, *the image of—under a recursive permutation*); hence, to establish the result claimed in §1, it suffices to exhibit a recursively enumerable set whose complement is regressive but not retraceable. This we shall now do.

THEOREM. *There exists a recursively enumerable set β such that β', while regressive, is nonretraceable.*

PROOF. Let $\{p_n(x)\}$ be the usual effective enumeration of the partial recursive functions of one variable, and let "P_n" denote the cumulative outcome of performing the first n steps in some fixed effective procedure P for generating precisely all correct equations $p_q(k)=r$. With each function p_n we associate a marker Λ_n. The set β, together with a recursive regressing function f for β', is constructed by stages as follows.

AT STAGE 0. Place (2, 1), (1, 3), (3, 0), and (0, 0) in f, attach Λ_0 to 3, and proceed to Stage 1.

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AT STAGE s, $s>0$. Let Λ_t be the marker of largest index which is attached to a number at the end of Stage $s-1$; we assume, as an inductive hypothesis (it will be easily seen, when the description of Stage s is complete, that the assumption persists from stage s to Stage $s+1$), that Λ_0 is one of the attached markers and that, indeed, *all* of $\Lambda_0, \Lambda_1, \cdots, \Lambda_t$ are attached at the end of Stage $s-1$. Now examine P_s. If P_s discloses that $p_0(3)=2$ and if 2 has not previously been placed in β, then (a) place 2 in β, (b) erase all markers Λ_j for $t\geqq j>0$, (c) place in β all numbers m, $m-1$, $m-2$ such that m was the position of a marker erased in (b), (d) attach Λ_1 to $n+2$, where n is the smallest number not previously placed in the domain of f, (e) place $(n+2, 1)$, $(n+1, n)$, and $(n, n+2)$ in f, and, finally, (f) proceed to Stage $s+1$. Otherwise: if $t\geqq 1$, check to see if P_s discloses that $p_1(k)=k-1$ where k is the number to which Λ_1 is attached and $k-1$ has not previously been placed in β. If so, then (a) place $k-1$ in β, (b) erase all markers Λ_j for $t\geqq j>1$, (c) place in β all numbers q, $q-1$, $q-2$ such that q was the position of a marker erased in (b), (d) attach Λ_2 to $m+2$, where m is the smallest number not previously placed in the domain of f, (e) place $(m+2, k-2)$, $(m+1, m)$, $(m, m+2)$ in f, and, finally (f) proceed to Stage $s+1$. But otherwise, *iterate the foregoing procedure until either* (i) *some alteration of marker positions and membership in* β *has been made and we have been ordered to go to Stage* $s+1$, *or* (ii) *all the positions of* $\Lambda_0, \cdots, \Lambda_t$ *at the end of stage* $s-1$ *have been scrutinized, with no such alterations having been authorized.* In case (ii), let u be the least number not yet assigned to the domain of f; attach Λ_{t+1} to $u+2$, and place in f the pairs $(u+1, u)$, $(u, u+2)$, and $(u+2, l^*)$, where l^* is either one or two less than the position h of Λ_t according as the number one less than h has not or has been placed in β; then go to Stage $s+1$.

This completes the description of the construction; clearly β (= the set of all n *placed in* β at some stage $s\geqq 0$) is recursively enumerable.

REMARK. At the end of Stage s, exactly the numbers $0, 1, 2, \cdots, 3(s+1)$ have been assigned to the domain of f.

PROOF. Trivial, by induction on s. The proof of the theorem is completed by a sequence of three straightforward lemmas.

LEMMA A. *Every marker* Λ_j, $j\geqq 0$, *eventually becomes permanently attached to some one number* n_j, *and* n_j *is of the form* $3k$, $k>0$.

PROOF. This is obtained by a routine induction on j. (In particular, we must have $n_0=3$.)

LEMMA B. β' *is regressed by* f.

PROOF. We first note that (as is clear from the construction and Lemma A), β' consists of the numbers 0, n_j (see Lemma A), n_j-2, and, *provided* $p_j(n_j)\neq n_j-1$, also n_j-1. From the description of Stage 0 we see that $f(0)=0, f(n_0)=f(3)=0, f(n_0-2)=f(1)=3$, and $f(n_0-1)=f(2)=1$. From the description of Stage s, we further have, for each $j>0$, that $f(n_j-2)=n_j$, $f(n_j-1)=n_j-2$, and $f(n_j)=n_{j-1}-2$ or $n_{j-1}-1$ according as $n_{j-1}-1$ is or is not in β. Thus, f does indeed regress β'.

LEMMA C. *β' is not retraceable.*

PROOF. Suppose that p_k retraces β'. Let s be a stage such that all Λ_t, $t\leqq k$, are in their final positions prior to s, all numbers $\leqq$ the position n_k of Λ_k which are destined for membership in β have already gone into β, and P_s discloses that $p_k(n_k)=r=$ the next smaller member of β'. If $r=n_k-1$, then n_k-1 must be placed in β at or before Stage s: contradiction. Hence $r=n_k-2$. But this is possible only if n_k-1 has previously been placed in β owing to discovery of $p_k(n_k)=n_k-1$: contradiction. Lemma C follows, and with it the theorem.

3. REMARKS. (a) By careful consideration of the proof of Proposition 5 in [1] as it applies to the sets constructed in Theorem T5 of [2], one sees how to show, without a "priority" construction as in §2, that retraceability is not always preserved by recursive permutations. The retraceable and regressive sets thus considered do not have recursively enumerable complements (by T4 of [2] and Proposition 10 of [1]); it does follow, however, that there are $2^{\aleph_0}$ nonhyperimmune retraceable sets whose retraceability is not preserved by all recursive permutations.

(b) A slight modification in the proof of the theorem leads to the result that *primitive* recursive permutations need not always preserve retraceability of complements of r.e. sets.

(c) We do not know yet exactly which r.e. degrees contain r.e. sets with regressive, nonretraceable complements; but we can show that any r.e. degree a for which $a'=0''$ does contain such a set.

REFERENCES

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