SMALL MODULES

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Introduction. In [1] and [5] a left A-module E is said to be small (or superfluous) in F if $E+H=F$ for any submodule H of F implies $H=F$. We define a left A-module S to be small if it is a small submodule of some module. In what follows we investigate some properties of small modules and prove the following theorems:

THEOREM. *A torsion module over a principal ideal domain is small if and only if the primary components are bounded.*

THEOREM. *If A is a discrete valuation ring with prime p, and G an A-module then the following conditions are equivalent*:

(1) *G is small,*

(2) *pG is small in G,*

(3) *G is the direct sum of a free module of finite rank and a bounded torsion module.*

The notation used in the following will be that of [2] and [3].[1]

LEMMA 1. *If E, F, and G are left A-modules such that $E\subset F\subset G$ and E is small in F then E is small in G.*

PROOF. Straightforward.

LEMMA 2. *If S is a small submodule of a left A-module F and S is contained in a direct summand E of F then S is small in E.*

PROOF. Straightforward.

THEOREM 1. *A left A-module F is small if and only if F is small in its injective envelope.*

PROOF. We will denote the injective envelope of a module F by $I(F)$.

If F is small in $I(F)$ then F is a small module by definition. Thus, suppose F is a small submodule of a left A-module H. Then $F\subset H \subset I(H)$, so by Lemma 1 F is small in $I(H)$. Assume $F+G=I(F)$ for some submodule G of $I(F)$. Since $I(F)$ is injective it is a direct summand of $I(H)$ and $F\subset I(F)$. Thus, by Lemma 2 F is small in $I(F)$.

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THEOREM 2. *Submodules, quotient modules and finite direct sums of small modules are small.*

PROOF. Straightforward.

COROLLARY. *The finite sum of small left A-modules which are submodules of a given module is a small module.*

If I is an infinite set it can be shown that $Z^{(I)}$ is not small in $Q^{(I)}$ and Z^N is not small in Q^N where Z is the additive group of integers, Q the additive group of rational numbers, and N the set of positive integers. But, Z is a small group. Moreover, $Z(p^\infty)$ is the sum of all its proper subgroups, their injective envelope, and each subgroup is small in $Z(p^\infty)$, but $Z(p^\infty)$ is not small.

We now show that a module over a principal ideal domain is small if and only if its torsion and torsion free parts are small.

LEMMA 3. *If E is a left A-module and $S\subset F$ are submodules of E such that S is small in E then F/S is small in E/S if and only if F is small in E.*

PROOF. Suppose $F+H=E$ for some submodule H of E. Then $F/S=(H+S)/S=E/S$, but F/S is small in E/S, hence $(H+S)/S=E/S$. Therefore, $H+S=E$. But, S is small in E, hence $H=E$. Thus F is small in E.

Conversely, assume $F/S+H/S=E/S$ for some submodule H containing S of E. Then $(F+H)/S=E/S$, hence $F+H=E$. But, F is small in E, hence $H=E$. Thus, F/S is small in E/S.

THEOREM 3. *If A is a left hereditary ring and the sequence of left A-modules, $0\to H\to G\to G/H\to 0$, is exact then H and G/H are small modules if and only if G is a small module.*

PROOF. Since H is small in $I(H)$, H is small in $I(G)$. Moreover, G/H is small in $I(G)/H$ since $I(G)/H$ is injective for A a hereditary ring. Hence, by Lemma 3 G is small in $I(G)$, therefore a small module.

Conversely, if G is small then H is small and G/H is a small module by Theorem 2.

COROLLARY. *A module over a principal ideal domain is small if and only if its torsion and torsion free parts are small.*

PROOF. A principal ideal domain is a hereditary ring.

LEMMA 4. *If G is a small module over a principal ideal domain then the torsion submodule, $T(G)$, is the only basic submodule of itself.*

PROOF. G is small in $I(G)$, hence $T(G)$ is small in $I(G)$. If B is a basic submodule of $T(G)$ then $T(G)/B$ is small in $I(G)/B$. But, $T(G)/B$ is divisible, hence $T(G)/B=0$ or $T(G)=B$.

LEMMA 5 (KULIKOV). *A primary module over a principal ideal domain has only one basic subgroup if and only if it is either divisible or bounded.*

PROOF. [3, Theorem 31.3].

THEOREM 4. *A torsion module T over a principal ideal domain is small if and only if the primary components of T are bounded.*

PROOF. Suppose T is small. T has a unique decomposition into its primary components and by Lemma 4 T is the only basic submodule of itself. Since T is not divisible by Lemma 5 the primary components are bounded.

Conversely, assume the primary components of T are bounded. Suppose $T+H=I(T)$ for some submodule H of $I(T)$. Then $T_p+H_p=I(T)_p$ where T_p, H_p, and $I(T)_p$ are the respective primary components. There exists an integer $N>0$ such that $p^NT_p=0$. Hence, $p^N(T_p+H_p)=p^NH_p=p^NI(T)_p=I(T)_p$. Then $H=\oplus_pH_p=\oplus_pI(T)_p=\oplus_pI(T_p)=I(T)$. Therefore, T is a small module.

LEMMA 6. *If A is a left hereditary ring then a left A-module F is small if and only if F has no nontrivial injective quotients.*

PROOF. Assume F is not a small module. Then there exists a submodule H or $I(F)$ such that $F+H=I(F)$ and $H\neq I(F)$. Then the sequence $0\rightarrow F\cap H\rightarrow F\rightarrow I(F)/H\rightarrow 0$ is exact and $I(F)/H$ is injective.

Conversely, if $F/H\neq 0$ is injective for some submodule H of F then F/H is a direct summand of $I(F)/H$. Thus, by Lemma 3 F is not small in $I(F)$.

LEMMA 7. *A small torsion free module over a principal ideal domain A has finite rank.*

PROOF. Suppose G is small, $rk(G)=\infty$, and $(x_i)_{i\in N}$ is a maximal linearly independent family of G. If K is the submodule generated by $(x_i)_{i\in N}$ then K is isomorphic to $A^{(N)}$ which is not small. Therefore, G is not small; contradiction.

If A is a principal ideal domain A_p will denote the localization of A at the prime p and $G_p=A_p\otimes_A G$ the localization of the A-module G.

LEMMA 8. *If A is a principal ideal domain and G an A-module then G is small if and only if G_p is small for all primes p.*

PROOF. Suppose G_p is small for all primes p and let J be an injective quotient of G. Then the exactness of the sequence $G \to J \to 0$ implies the sequence $G_p \to J_p \to 0$ is exact for every prime p. Moreover, J_p is injective. But, $J_p = 0$ for every prime p if and only if $J = 0$ ([2], p. 82). Hence, the conclusion follows from Lemma 6. (Note that this proof holds for any commutative ring.)

Suppose that G is small and $J = G_p/H \neq 0$ is an injective quotient. We may take J to be torsion. There is an $E \subset G$ such that $H = E_p$, and we have $J = G_p/E_p = (G/E)_p$. It follows that G/E is torsion with p-primary component J, so J is a quotient of G. Since J is automatically A-injective G is not small; contradiction.

THEOREM 5. *Let A be a discrete valuation ring with prime p, and let G be an A-module. The following conditions are equivalent*:

(1) *G is small*,

(2) *pG is small in G*,

(3) *G is the direct sum of a free module of finite rank and a bounded torsion module.*

PROOF. If $H \subset G$ then G/H is injective $\Leftrightarrow p(G/H) = G/H \Leftrightarrow H + pG = G$. Thus, (1)$\Leftrightarrow$(2) follows from Lemma 6.

(3)$\Rightarrow$(1) follows from Theorem 2, Theorem 4, and the fact that A is small.

(1)$\Rightarrow$(3). By Theorem 4 the torsion part of G is bounded. Using the corollary to Theorem 6 we reduce the problem to the torsion free case. By Lemma 7 G then has finite rank. We want to show that G is finitely generated, so suppose it is not. Choose F a free submodule of G such that G/F is torsion. G/F is not finitely generated since F is and G is not. Let $\hat{A}$ denote the completion of A. By Theorem 20 [**4**] $\hat{A} \otimes G$ is the direct sum of a free and a divisible module. Since $G/F = \hat{A} \otimes (G/F) = (\hat{A} \otimes G)/(\hat{A} \otimes F)$ is not finitely generated neither is $\hat{A} \otimes G$. Hence, $\hat{A} \otimes G$ has a nontrivial divisible part so, since $\hat{A} \otimes F$ is free, it follows that G/F contains a nontrivial divisible module. This contradicts the assumed smallness of G.

COROLLARY. *If A is a principal ideal domain and if G is an A-module then G is small if and only if G is locally a free module of finite rank plus a bounded torsion module.*

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AN ADDITION TO ADO'S THEOREM[1]

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The main purpose of this note is to point out the following strengthened (with respect to the nilpotency property) form of the theorem on the existence of a faithful finite-dimensional representation of a finite-dimensional Lie algebra.

THEOREM 1. *Let L be a finite-dimensional Lie algebra over an arbitrary field, and let α denote the adjoint representation of L. There exists a faithful finite-dimensional representation ρ of L such that $\rho(x)$ is nilpotent for every element x of L for which $\alpha(x)$ is nilpotent.*

For the suggestion that this nilpotency property of ρ might be secured I am indebted to Leonard Ross who used the characteristic 0 case of Theorem 1 in his proof of Ado's Theorem for graded Lie algebras (Thesis, *Cohomology of graded lie algebras*, University of California, Berkeley, 1964).

In the case of characteristic 0, it is known that there exists a faithful finite-dimensional representation of L whose restriction to the maximum nilpotent ideal of L is nilpotent [**1**, pp. 202–203]. Hence, in order to establish Theorem 1 in the case of characteristic 0, it suffices to make the following observation:

Let L be a finite-dimensional Lie algebra over a field of characteristic 0, *and let M be a finite-dimensional L-module on which the maximum nilpotent ideal N of L is nilpotent. Let x be an element of L whose adjoint image $\alpha(x)$ is nilpotent. Then x is nilpotent on M.*

PROOF. Write $L=S+R$, where R is the radical of L and S is a semisimple subalgebra of L. Accordingly, write $x=s+r$, with s in S and r in R. Since $\alpha(x)$ is nilpotent, it is clear that the adjoint representation of S sends s onto a *nilpotent* derivation of S. Since S

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