A NOTE ON INVARIANT INTEGRALS ON LOCALLY COMPACT SEMIGROUPS

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1. Introduction. An integral on a locally compact (Hausdorff) semigroup S is a nontrivial, positive, linear functional I on the space $C(S)$ of continuous real-valued functions on S with compact supports. If S satisfies the condition

(#) for each compact set $K \subset S$ and each element $a \in S$, the set

$$Ka^{-1} = \{x \in S \mid xa \in K\}$$

is compact; then, whenever $f \in C(S)$ and $a \in S$, the function f_a defined by $f_a(x) = f(xa)$ is also in $C(S)$.

In this case, an integral I on S is called *right invariant* provided $I(f) = I(f_a)$ for all $f \in C(S)$ and all $a \in S$. A regular Borel measure μ on S is called *r^*-invariant* if $\mu(Ba^{-1}) = \mu(B)$ for all Borel sets B and all $a \in S$.

J. H. Michael [7] introduced the above concept of an invariant integral[1] and proved that if S contains a unique minimal left ideal and satisfies some additional conditions (see [7]) then S admits a right invariant integral. P. S. Mostert [8] then pointed out that Michael's conditions could be weakened considerably and he also gave a much shorter proof. In this note we prove the following:

THEOREM 1. *Let S be a locally compact semigroup satisfying the condition (#). Then the following are equivalent.*

(a) *S admits a right invariant integral.*

(b) *S admits an r^*-invariant measure.*

(c) *S contains a unique minimal left ideal (which is necessarily closed).*

Thus we obtain a complete generalization of the theorem of W. G. Rosen [10] on the existence of invariant means on compact semigroups. In addition, we make some remarks concerning the existence and structure of r^*-invariant measures on semigroups which do not necessarily satisfy (#).

For matters concerned with integrals and measures on locally compact spaces, we follow Bourbaki [2, Chapter III], and Hewitt and Ross [6, §11]. In particular, the family of Borel sets is defined to be the σ-algebra generated by the family of all open sets.

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[1] Our condition (#) is a weaker form of Michael's condition (A).

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2. **Right-invariant integrals on locally compact left groups.** In this section we establish the existence and analyze the structure of right-invariant integrals on locally compact left groups. Recall that a left group is a semigroup S which is left simple ($Sa=S$ for all $a\in S$) and right cancellative. It is well known (see [**4**, p. 38]) that the following conditions on a semigroup S are equivalent:

(i) S is a left group.

(ii) S is left simple and contains an idempotent.

(iii) S is isomorphic to a direct product $E\times G$ where E is a left zero semigroup and G is a group.

In regard to the latter, we recall that each idempotent acts as a right identity in S and that a realization of S as a direct product is accomplished by taking E to be the set of all idempotents in S and letting $G=e_0S$ where e_0 is a fixed (but arbitrary) element of E. The mapping $(e, e_0x)\rightarrow ex$ is then an isomorphism of $E\times G$ onto S. If, in addition, S is a locally compact semigroup, then this mapping (call it ϕ) is topological. In fact, we have the following structure theorem.

THEOREM 2. *Let S be a locally compact left group. Then S is topologically isomorphic to a direct product $E\times G$ where E is a locally compact left zero semigroup and G is a locally compact topological group.*

This theorem is a special case of the theorem proved by Mostert in [**8**]; we omit the proof.

Now we apply the above structure theorem to establish the existence and structure of right-invariant integrals on a locally compact left group S. Note that each element of S has a right inverse with respect to some right identity in S and hence the right translates $r_a(x)=xa$ are homeomorphisms of S. Thus S satisfies the condition (#).

THEOREM 3. *Let S be a locally compact left group. Then S admits a right-invariant integral. Moreover, each right-invariant integral I on S can be obtained as follows:*

Write $S=E\times G$ as in Theorem 2. Then

$$I(f) = \int_S f\, d(\nu\times\lambda) \tag{1}$$

where ν is a positive regular Borel measure on E and λ is a right Haar measure on G.

Proof. Writing (1) as an iterated integral, it is easy to see that any such measure gives rise to a right-invariant integral on S. See [1, Theorem 1.2], for the details of a similar argument.

Suppose, conversely, that I is a right-invariant integral on S. We observe that I is completely determined by its values on functions of the form $h(e, g) = \psi(e)\phi(g)$ where $\psi \in C(E)$ and $\phi \in C(G)$; this is because linear combinations of functions of this type are "riche" in $C(E \times G)$ (see [2, p. 55 and p. 89]). Choose fixed nonnegative functions $\psi_0 \in C(E)$ and $\phi_0 \in C(G)$ such that $I(\psi_0\phi_0) = 1$, and let $H(\phi) = I(\psi_0\phi)$ for all $\phi \in C(G)$. Then H is a positive, right-invariant integral on G; that is, H is a right Haar integral on G. Next we note that for each fixed $\psi \in C(E)$, the mapping $\phi \rightarrow I(\psi\phi)$ is a right-invariant Radon measure (see [2, p. 50]) on G. Hence, for each ψ, there exists a real number $J(\psi)$ such that

$$I(\psi\phi) = J(\psi)H(\phi) \tag{2}$$

for all $\phi \in C(G)$. It is easy to see that J is an integral on E. Equation (2) also shows that I is the product $J \times H$ of the integrals J and H (see [2, p. 89]). Finally, the formula (1) now follows from the well-known relationship between integrals and measures.

We remark that the measure $\mu = \nu \times \lambda$ is both r^*-invariant and right-invariant:

$$\mu(Ba^{-1}) = \mu(B) = \mu(Ba) \quad \text{for all Borel sets } B \subset S \text{ and all } a \in S.$$

3. **Proof of Theorem 1.** Throughout this section S will denote a locally compact semigroup satisfying the condition (#). It is convenient to introduce the following weak form of condition (b).

(b′) S admits a positive regular Borel measure μ satisfying $\mu(Ua^{-1}) = \mu(U)$ for all open sets $U \subset S$ and all $a \in S$. We will prove Theorem 1 by establishing the implications (a)⇒(b′)⇒(c)⇒(a) and (c)⇒(b).

(a)⇒(b′). Let I be a right-invariant integral on S and let μ be the corresponding regular Borel measure on S. For open sets U, we have $\rho(U) = \sup \{I(f) \mid f \in C(S),\ 0 \leqq f \leqq \chi_U\}$; thus $\mu(Ua^{-1}) \geqq \sup\{I(f_a) \mid 0 \leqq f \leqq \chi_U\} = \mu(U)$.

Let K be a compact set and $\epsilon > 0$. By regularity of μ, there exists an open set U containing K such that $\mu(U) < \mu(K) + \epsilon$. Since S is locally compact, there exists a function $f \in C(S)$ such that $f \colon S \rightarrow [0, 1]$, $f(K) = 1, f(S \sim U) = 0$. Then $\mu(Ka^{-1}) \leqq \int_S f(ta)d\mu(t) = \int_S f(t)d\mu(t) \leqq \mu(U) < \mu(K) + \epsilon$. Thus $\mu(Ka^{-1}) \leqq \mu(K)$ for all $a \in S$. Note that if K is compact then Ka is compact and, since $K \subset (Ka)a^{-1}$, it follows that $\mu(K) \leqq \mu((Ka)a^{-1}) \leqq \mu(Ka)$.

Now let U be open and let K be a compact set contained in Ua^{-1}.

Then Ka is compact and contained in U, whence we have $\mu(K) \leqq \mu(Ka) \leqq \mu(U)$. But, again by regularity, $\mu(Ua^{-1}) = \sup\{\mu(K) \mid K$ compact, $K \subset Ua^{-1}\}$, and it follows that $\mu(Ua^{-1}) \leqq \mu(U)$. This completes the proof.

(b′)⇒(c). Let μ be a measure on S satisfying the condition (b′), and let F be the support of μ. If L is a nonempty closed left ideal in S and $a \in L$, then $\mu(S \sim L) = \mu((S \sim L)a^{-1}) = \mu(\emptyset) = 0$, whence it follows that $F \subset L$. Thus $L_0 = \bigcap \{L \mid L$ is a closed left ideal$\}$ is nonempty and contains F. Clearly L_0 is the unique minimal closed left ideal in S. We will show that $L_0 \subset L_0 a$ for all $a \in S$, whence it follows that L_0 is the unique minimal (algebraic) left ideal.

Let $a \in S$ and $b \in L_0$. Since $L_0 a$ is a left ideal we must have $L_0 \subset \mathrm{Cl}(L_0 a)$; thus there exists a net $\{t_\alpha\}$ in L_0 such that $\lim_\alpha t_\alpha a = b$. If K is a compact neighborhood of b, then $\{t_\alpha\}$ is eventually in the compact set $Ka^{-1} \cap L_0$, and hence there is a subnet $\{t_\beta\}$ which converges to a point t_0 in L_0. Then $t_0 a = \lim_\beta t_\beta a = b$, and so $b \in L_0 a$.

(c)⇒(a) and (c)⇒(b). Let L_0 be the unique minimal left ideal in S. We note that $L_0 a$ is minimal for each $a \in S$ (see [**3**, Lemma 2.1]), and thus $L_0 a = L_0$. In particular, L_0 is also a right ideal. We will show first that L_0 is closed, and then that L_0 is a left group.

Let $a \in S$ and $b \in \bar{L}_0$. Since $L_0 = L_0 a$, there is a net $\{t_\alpha\}$ in L_0 such that $t_\alpha a \to b$. If K is a compact neighborhood of b, then $\{t_\alpha\}$ is eventually in Ka^{-1}, and so there is a subnet $\{t_\beta\}$ which converges to some point t_0 in $\bar{L}_0$. Then $t_0 a = \lim_\beta t_\beta a = b$, and so $b \in \bar{L}_0 a$. We have shown that $\bar{L}_0 a = \bar{L}_0$ for all $a \in S$. If $a \in L_0$, then we have $\bar{L}_0 = \bar{L}_0 a \subset L_0$; hence L_0 is closed. Now let a_0 be a fixed element of L_0. Then $\{t \in L_0 \mid t a_0 = a_0\} = a a_0^{-1} \cap L_0$ is a nonempty, compact, subsemigroup of L_0 and hence contains at least one idempotent (see [**9**, Lemma 4]). Thus L_0 is a left group.

Now let $\sigma = \nu \times \lambda$ be a product measure on L_0 as in Theorem 3, and let $\mu(B) = \sigma(B \cap L_0)$ for all Borel sets $B \subset S$. Let e be a fixed idempotent in L_0. Then, $ea \in L_0$ for each $a \in S$, and

$$\begin{aligned} Ba^{-1} \cap L_0 &= \{x \in L_0 \mid xa \in B\} = \{x \in L_0 \mid x(ea) \in B\} \\ &= \{x \in L_0 \mid x(ea) \in B \cap L_0\} = (B \cap L_0)(ea)^{-1} \cap L_0 \end{aligned}$$

for all $B \subset S$. Since σ is r^*-invariant on L_0, it follows that

$$\begin{aligned} \mu(Ba^{-1}) &= \sigma(Ba^{-1} \cap L_0) = \sigma((B \cap L_0)(ae)^{-1} \cap L_0) \\ &= \sigma(B \cap L_0) = \mu(B) \end{aligned}$$

for all Borel sets $B \subset S$. Similarly, for each $f \in C(S)$ we have

$$\int_S f(xa)\,d\mu(x) = \int_{L_0} f(xa)\,d\sigma(x) = \int_{L_0} f(x(ea))\,d\sigma(x)$$
$$= \int_{L_0} f(x)\,d\sigma(x) = \int_S f(x)\,d\mu(x).$$

Hence $I(f) = \int_S f d\mu$ is a right invariant integral on S.

4. **Remarks on r^*-invariant measures.** In this section S will denote a locally compact semigroup not necessarily satisfying (#). We note that if S contains a closed right ideal F which is a left group, then any product measure on F of the type in Theorem 3 gives rise to an r^*-invariant measure on S. The author conjectures that every r^*-invariant measure arises in this way, and the remainder of this section is devoted to some observations which lend some support to this conjecture.

One can easily see (as in the proof of (b')$\Rightarrow$(c)) that if S admits an r^*-invariant measure μ, then S contains a unique minimal *closed* left ideal L_0 which must necessarily contain the support of μ. Furthermore, the support of μ has the following tantalizing property.

PROPOSITION 1. *Suppose that S admits an r^*-invariant measure μ, and let F be the support of μ. Then F is a closed right ideal in S and* $\mathrm{Cl}(Fa) = F$ *for all* $a \in S$.

PROOF. Let $a \in S$, $x \in F$, and let U be an (open) neighborhood of xa. Then $x \in Ua^{-1}$ and, since $x \in F$, it follows that $\mu(U) = \mu(Ua^{-1}) > 0$. Thus $xa \in F$, and so F is a right ideal. Now let $a \in S$, $b \in F$, and let V be an (open) neighborhood of b. Then $\mu(Va^{-1}) = \mu(V) > 0$, and so $Va^{-1} \cap F \neq 0$. This implies that $V \cap Fa \neq 0$, and we conclude that $b \in \mathrm{Cl}(Fa)$.

We conjecture that F must in fact be a left group and thus that μ is a measure of the type described above, but we have been able to prove this only in the case that S is discrete.

PROPOSITION 2. *Let S be a discrete semigroup. Then S admits an r^*-invariant measure if and only if S contains a right ideal which is a left group.*

PROOF. We have only to prove the necessity. Suppose then that S admits an r^*-invariant measure μ and let F be the support of μ. By Proposition 1, we know that F is a left simple right ideal. It remains only to show that F contains an idempotent.

For a fixed element $a \in F$, let $A = \{x \in F \mid xa = a\}$. Then A is nonvoid and $0 < \mu(A) = \mu(\{a\}) < \infty$. Note also that if $x \in A$ and

$y \in Ax^{-1} \cap F$ then $ya = y(xa) = (yx)a = a$, and so $y \in A$. Thus $Ax^{-1} \cap F \subset A$ for all $x \in A$ and it follows that the restriction of μ to A is an r^*-invariant measure on A. An application of the Recurrence Theorem (see Halmos, *Lectures in ergodic theory*, Chelsea, New York, 1956; p. 10) shows that each element of A must have finite order. We complete the proof by recalling that each finite semigroup contains an idempotent.

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