## SHORTER NOTES

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# ON A THEOREM OF MOTZKIN CONCERNING POWER SERIES WITH PERIODIC GAPS[1]

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**1. Introduction.** Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence 1. If $a_n = 0$ for all $n \equiv r \pmod{m}$, we say that $f(z)$ has the gap $r \pmod{m}$. Here $r$ and $m$ are fixed integers with $m > 0$. Now suppose $f(z)$ has $g$ distinct gaps $r_1, \cdots, r_g \pmod{m}$. These gaps are called *strongly distinct* if for any divisor $d$ of $m$, the number of incongruent $r_i \pmod{d}$ is $\geqq \min(g, d)$. Thus for example the gaps 1, 2, 4, 6 (mod 8) are distinct but not strongly distinct.

Motzkin [1] has shown that if $g \leqq 3$, and if $f(z)$ has $g$ strongly distinct gaps (mod $m$), then $f(z)$ has at least $g+1$ singularities on $|z| = 1$. In [2] he showed that when $g = 4$ or $g \geqq 6$, the number $s$ of singularities on $|z| = 1$ can be less than $g+1$, but left unanswered the question of determining the minimum value of $s$ for given $g$. In this paper it is shown that the answer to this problem is $\min(g+1, 4)$ for every $g \geqq 0$. In §2 we prove that $s \geqq \min(g+1, 4)$, and in §3 we construct examples with $s = \min(g+1, 4)$.

2. **Proof that** $s \geqq \min(g+1, 4)$. Since Motzkin has already proved this for $g \leqq 3$, we can assume $g \geqq 4$; it must then be shown that $s \geqq 4$. One might at first think that this could be done by choosing a subset of 3 strongly independent gaps from the given set of $g$ gaps, and then applying Motzkin's result. But this is not possible in general, as can be seen by considering the example where the gaps are the residue classes 0, 2, 8, 10, 11 and 14 (mod 15).

The proof for $g \geqq 4$ is, however, patterned after that of [1] for $g = 3$. The function $f(z)$ has at least one singularity on $|z| = 1$, say at $\alpha$. Suppose the given gaps are $r_1, r_2, \cdots, r_g \pmod{m}$, and let $\epsilon$ be a primitive $m$th root of unity. It suffices to show that at least 4 of the $m$ functions $f(\epsilon^k z)$ $(k = 0, 1, \cdots, m-1)$ are singular at $z = \alpha$. We suppose that only three of them, say $f(z), f(\epsilon^a z)$, and $f(\epsilon^b z)$ are singular

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at $\alpha$, and obtain a contradiction as follows (the proof would be even easier if only one or two of the functions $f(\epsilon^k z)$ were singular at $\alpha$; we omit these cases here). The fact that $f(z)$ has the gaps $r_1, \cdots, r_g$ (mod $m$), is expressed by the identities

$$\sum_{k=0}^{m-1} \epsilon^{-kr_l} f(\epsilon^k z) \equiv 0 \qquad (l = 1, \cdots, g).$$

These linear equations cannot be solved for the functions $f(z)$, $f(\epsilon^a z)$, $f(\epsilon^b z)$ in terms of the remaining functions $f(\epsilon^k z)$, for the latter are regular at $\alpha$. Hence the $g \times 3$ matrix

$$M = \begin{bmatrix} 1 & \epsilon^{-r_1 a} & \epsilon^{-r_1 b} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 1 & \epsilon^{-r_g a} & \epsilon^{-r_g b} \end{bmatrix}$$

has rank $<3$. Since its elements have absolute value 1, it follows (cf. [1], p. 99) that either the rows or the columns of $M$ can be divided into two sets $M_1$ and $M_2$ such that the matrices $M_1$ and $M_2$ have rank 1. Suppose first that it is the rows which can be so divided. If the $i$th and $j$th rows are proportional, they are equal, so

$$\epsilon^{r_i a} = \epsilon^{r_j a} \quad \text{and} \quad \epsilon^{r_i b} = \epsilon^{r_j b}.$$

Hence

$$(r_i - r_j)a \equiv (r_i - r_j)b \equiv 0 \pmod{m}.$$

It follows that $(a, m) = d > 1$, and that $r_i \equiv r_j$ (mod $m/d$). Hence the $r_i$ fall into at most two residue classes (mod $m/d$), corresponding respectively to the rows of $M_1$ and $M_2$. By the definition of strongly distinct gaps, we must have $m/d = 2$, i.e. $(a, m) = m/2$. This implies that $a = m/2$. Reasoning in the same way with the congruences $(r_i - r_j)b \equiv 0$ (mod $m$), we find that $b = m/2$, a contradiction to the fact that $a \neq b$.

If it is the columns of $M$ which can be divided into the two sets $M_1$, $M_2$ as described above, then there must in particular be two proportional columns. If, the 1st and 2nd columns are proportional, then $\epsilon^{(r_i - r_j)a} = 1$, so $(r_i - r_j)a \equiv 0$ (mod $m$) for all $i$, $j$. This leads to the contradiction $r_i - r_j \equiv 0$ (mod $m/d$), where $d = (a, m)$. A similar contradiction arises by supposing proportionality of any two columns.

It should be noted that in the above proof the full force of the hypothesis of strong distinctness was not used. It would suffice that for any divisor $d > 2$ of $m$, there are at least three incongruent $r_i$ (mod $d$), and in case $2 \mid m$, there are both odd and even $r_i$.

**3. Examples with $s=\min(g+1, 4)$.** Let $m=pq$, where $p$ and $q$ are distinct primes, and let $\epsilon$ be a primitive $pq$th root of unity. Put

$$f(z) = (1 - z^{pq})^{-1} \prod_{\mu=1}^{q-2} (z^p - \epsilon^{\mu p}) \prod_{\nu=1}^{p-2} (z^q - \epsilon^{\nu q}) = (1 - z^{pq})^{-1}\Pi_1\Pi_2.$$

The singularities of $f(z)$ are at those points $\epsilon^k$ which are not zeros of $\Pi_1$ or $\Pi_2$. The factors of $\Pi_1$ are relatively prime in pairs, and the same holds for the factors of $\Pi_2$. Moreover the polynomials $z^p-\epsilon^{\mu p}$ and $z^q-\epsilon^{\nu q}$ have exactly one zero in common. Hence $\Pi_1\Pi_2$ has $p(q-2)+q(p-2)-(q-2)(p-2)=pq-4$ zeros. Thus $f(z)$ has exactly four singularities, all on $|z|=1$.

To determine the gaps of $f(z)$, we expand $(1-z^{pq})^{-1}$ in a geometric series, getting

$$f(z) = \prod_{\mu=1}^{q-2} (z^p - \epsilon^{\mu p}) \prod_{\nu=1}^{p-2} (z^q - \epsilon^{\nu q})(1 + z^{pq} + z^{2pq} + \cdots).$$

From this expression it follows that the coefficients $a_n$ in the expansion $f(z)=\sum_{n=0}^{\infty} a_n z^n$ are zero unless $n\equiv ap+bq \pmod{pq}$, where $0\leqq a\leqq q-2$, $0\leqq b\leqq p-2$. Hence $f(z)$ has (at least) $p+q-1$ distinct gaps (mod $pq$), namely the residue classes $(q-1)p+bq$ and $ap+(p-1)q$, where $0\leqq a\leqq q-1$, $0\leqq b\leqq p-1$. We shall prove that these gaps are strongly distinct. For this purpose it suffices to show that they represent every residue class $c$ (mod $p$) and every residue class $d$ (mod $q$). The congruence $(q-1)p+bq\equiv c \pmod{p}$ is equivalent to $bq\equiv c \pmod{p}$, and this has a unique solution $b$ (mod $p$), since $(p, q)=1$. Similarly we can solve $ap+(p-1)q\equiv d \pmod{q}$ for $a$.

Next note that the residues $ap+(p-1)q$ are all $\equiv -q \pmod{p}$, while the residues $(q-1)p+bq$ are all $\equiv -p \pmod{q}$. Hence if we delete the gap $pq-p-q$, the remaining gaps are still strongly distinct. Deletion of such a gap can be accomplished without affecting the number of singularities on $|z|=1$ by adding to $f(z)$ an appropriate entire function (such as $z^{pq-p-q}\exp z^{pq}$), or an appropriate rational function (such as $z^{pq-p-q}(2-z^{pq})^{-1}$).

Thus we have constructed functions with $p+q-1$ or $p+q-2$ strongly distinct gaps and exactly 4 singularities on $|z|=1$. If Goldbach's conjecture is true, every integer $\geqq 3$ is of the form $p+q-1$ or $p+q-2$. Together with the example $(1-z^{g+1})^{-1}$ for $g=0$, 1, or 2, this would imply that the inequality $s\geqq\min(g+1, 4)$ is best possible. To see this without resort to Goldbach's conjecture, we take $q=2$ in the above construction. After deleting the residue class $pq-p-q=p-2$, as already explained, we are left with $p+q-2=p$ gaps, namely the residue classes $2p-2$ and $p+2b$ (mod $2p$), where $0 \leqq b \leqq p-2$. These numbers represent every residue class (mod $p$) exactly once. Since the first of them is even and the others are all odd, we can delete as many of the gaps $p+2b$ as we please without destroying the property of strong distinctness. Thus by adding entire (or rational) functions to $f(z)$ as above, we can construct examples of functions with $s=4$ and $g$ any integer in the range $0 \leqq g \leqq p+1$. This clearly does what is required.

The above example can be modified so that the coefficients $a_n$ in the expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$ are all real. For example, if $q>2$, replace the expression $\prod_{\mu=1}^{q-2} (z^p - \epsilon^{\mu p})$ by $(z^p-1) \prod_{\mu=2}^{q-3} (z^p - \epsilon^{\mu p})$, and if $p>2$, replace $\prod_{\nu=1}^{p-2} (z^q - \epsilon^{\nu q})$ by $(z^q-1) \prod_{\nu=2}^{p-3} (z^q - \epsilon^{\nu q})$. These polynomials have real coefficients, and the analysis of gaps and singularities is essentially the same as before.

## References

1. T. S. Motzkin, *Bemerkung über Singularitäten gewisser mit Lücken behafteter Potenzreihen*, Math. Ann. **109** (1933), 95–100.

2. ———, *Power series with gaps*, Proc. Amer. Math. Soc. **11** (1960), 875.

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