ON SECTIONAL CURVATURES AND CHARACTERISTIC OF HOMOGENEOUS SPACES

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Let X be a compact orientable Riemannian manifold of even dimension n. The generalized Gauss-Bonnet theorem [1] states that the Euler-Poincaré characteristic of X is

$$\chi(X) = \frac{2}{c_n}\int_X \gamma_n \omega \tag{1}$$

where c_n is the volume of the Euclidean unit n-sphere, γ_n the nth sectional curvature (see the definition (2) below) and ω the volume element of the Riemannian structure of X. It is a still open question, whether the fact that the usual sectional curvature (second order sectional curvature) γ_2 has a constant sign for all plane sections, has some implications on the sign of γ_n. Such results would give interesting applications via the generalized Gauss-Bonnet theorem. A known result in this direction is Milnor's theorem (see [2, Theorem 5]), stating that for $n=4$ the Euler-Poincaré characteristic is positive, if γ_2 is always positive or always negative.

We shall consider the class of Riemannian manifolds arising by division of a compact Lie group G by a closed subgroup H and equipment of the quotient G/H with the invariant Riemannian metric g induced by a bi-invariant metric $\tilde{g}$ on G. Consider the orthogonal decomposition

$$\boldsymbol{G} = \boldsymbol{H} \oplus M$$

with respect to g, turning G/H into a reductive homogeneous space. We shall make the assumption that G/H is locally symmetric, i.e. $[M, M] \subset \boldsymbol{H}$. Let $n = \dim G/H$. With these notations we shall prove the following.

THEOREM. *Let p be any even integer with $0 < p \leqq n$. Then the pth sectional curvature γ_p is nonnegative.*

REMARK. For $p=2$ this was proved in [6] even without the assumption of the local symmetry of G/H, and follows also at once from the formulae in [5]. As remarked in [7], the proof of Milnor's result in [2] shows that $\gamma_2 \geqq 0$ implies $\gamma_4 \geqq 0$.

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COROLLARY [3]. *Let the situation be as before. Then the Euler-Poincaré characteristic of G/H is nonnegative.*

PROOF. Without loss of generality we can assume that H is connected. Hence G/H is orientable. We only have to consider the case when the dimension n of G/H is even. Then the Gauss-Bonnet theorem yields the desired result.

This answers, for the class of spaces considered, the question raised in [6, p. 13, line 5].

REMARK. Suppose the dimension n of G/H to be even. Then by homogeneity, the sectional curvature γ_n is seen to be constant [7]. Therefore $\chi(G/H)$ and γ_n are either both positive or both zero.

We recall the definition of the sectional curvatures of a Riemannian manifold X of (not necessarily even) dimension n (see [7]). Let p be an even integer with $0<p\leqq n$, $x\in X$ and $P\subset T_x(X)$ a p-plane at x. Let $X_1, \cdots, X_p$ be any orthonormal base of P and R the curvature tensor at X of the Riemannian metric $\langle\ ,\ \rangle$. Then the pth sectional curvature of the p-plane P is given by

$$\gamma_p(x; P) = \frac{(-1)^{P-2}}{2^{p/2}\cdot p!}\sum_{\sigma,\tau}\epsilon(\sigma)\epsilon(\tau)\langle R(X_{\sigma_1}, X_{\sigma_2})X_{\tau_1}, X_{\tau_2}\rangle\cdots \tag{2}$$
$$\cdots\langle R(X_{\sigma_{p-1}}, X_{\sigma_p})X_{\tau_{p-1}}, X_{\tau_p}\rangle.$$

Here the sum ranges over all permutations σ, τ of the set $\{1, \cdots, p\}$ and $\epsilon(\sigma)$, $\epsilon(\tau)$ are the signs of the permutations σ, τ respectively. For $p=2$, formula (2) is the usual expression

$$\gamma_2(x; P) = -\langle R(X_1, X_2)X_1, X_2\rangle \tag{3}$$

in view of the skew-symmetry of the operator $R(X_1, X_2)$ with respect to $\langle\ ,\ \rangle$.

We now turn to the case $X=G/H$ considered in the theorem. As G/H is supposed to be locally symmetric in its canonical reductive structure, the canonical connection [5] is the Riemannian connection of $\tilde{g}$. Let $\langle\ ,\ \rangle$: $\mathbf{G}\times\mathbf{G}\to R$ denote the restriction of g to $\mathbf{G}$ and let x_0 be the point of G/H corresponding to the unit e of G. $T_{x_0}(G/H)$ is identified with the orthogonal complement M of $\mathbf{H}$ in $\mathbf{G}$. Then we have the following known

LEMMA. *Let R be the curvature tensor of $\tilde{g}$ in x_0. Then for X, Y, V, $W\in T_{x_0}(G/H)$ we have*

$$\langle R(X, Y)V, W\rangle = -\langle[X, Y], [V, W]\rangle.$$

PROOF. By [5, Theorem 10.3], we have in view of the preceding remarks

$$R(X, Y)V = -[[X, Y], V].$$

Note that $[X, Y] \in \boldsymbol{H}$, as we have supposed G/H to be locally symmetric. Now $\langle\, ,\rangle$ is invariant under the adjoint representation of H, so that

$$\langle [[X, Y], V], W\rangle = \langle [X, Y], [V, W]\rangle.$$

This proves the lemma.

The theorem is now a consequence of the following

PROPOSITION. *Let the situation be as above. Let $P \subset T_{x_0}(M)$ be a p-plane, p an even integer with $0 < p \leqq n = \dim (G/H)$ and $X_1, \cdots, X_r$ ($r = \dim G$) an orthonormal basis of $\boldsymbol{G}$ such that the first p vectors lie in P and the last $r-n$ vectors in $\boldsymbol{H}$. Then the pth sectional curvature is given by*

$$\gamma_p(x_0; P) = \frac{1}{2^{p/2}\cdot p!} \cdot \sum_{k_1, \cdots, k_{p/2}} \left(\sum_{\sigma} \epsilon(\sigma) \langle [X_{\sigma_1}, X_{\sigma_2}], X_{k_1}\rangle \cdots \langle [X_{\sigma_{p-1}}, X_{\sigma_p}], X_{k_{p/2}}\rangle \right)^2 \tag{4}$$

where σ runs through the permutations of $\{1, \cdots, p\}$, $\epsilon(\sigma)$ is the sign of σ, and $(k_1, \cdots, k_{p/2})$ runs through the $p/2$-tuples of integers k_i with $r-n < k_i \leqq r$ for $i = 1, \cdots, p/2$.

PROOF. By (2), we have in virtue of the lemma

$$\gamma_p(x_0; P) = \frac{1}{2^{p/2}\cdot p!} \sum_{\sigma,\tau} \epsilon(\sigma)\epsilon(\tau) \langle [X_{\sigma_1}, X_{\sigma_2}], [X_{\tau_1}, X_{\tau_2}]\rangle \cdots \cdots \langle [X_{\sigma_{p-1}}, X_{\sigma_p}][X_{\tau_{p-1}}, X_{\tau_p}]\rangle.$$

We write $c_{\alpha\beta}^{\gamma}$ for $\langle [X_\alpha, X_\beta], X_\gamma\rangle$, so that $[X_\alpha, X_\beta] = \sum_{\gamma=1}^{r} c_{\alpha\beta}^{\gamma} X_\gamma$ (the c's are the structural constants of $\boldsymbol{G}$). Using $[M, M] \subset \boldsymbol{H}$ and the orthonormality of the base we obtain then

$$\langle [X_{\sigma_1}, X_{\sigma_2}], [X_{\tau_1}, X_{\tau_2}]\rangle = \sum_{k_1=r-n+1}^{r} c_{\sigma_1\sigma_2}^{k_1} c_{\tau_1\tau_2}^{k_1}$$

and similar expressions for the other terms in (5). Hence

$$\gamma_p(x_0; P) = \frac{1}{2^{p/2}\cdot p!} \sum_{\sigma,\tau} \epsilon(\sigma)\epsilon(\tau) \left(\sum_{k_1=r-n+1}^{r} c_{\sigma_1\sigma_2}^{k_1} c_{\tau_1\tau_2}^{k_1} \right) \cdots \cdots \left(\sum_{k_{p/2}=r-n+1}^{r} c_{\sigma_{p-1}\sigma_p}^{k_{p/2}} c_{\tau_{p-1}\tau_p}^{k_{p/2}} \right). \tag{6}$$

The $\sum_{\sigma,\tau}$ in (6) can be rewritten as

$$\sum_{k_1,\cdots,k_{p/2}=r-n+1}^{r}\left(\sum_\sigma \epsilon(\sigma)c_{\sigma_1\sigma_2}^{k_1}\cdots c_{\sigma_{p-1}\sigma_p}^{k_{p/2}}\right)\left(\sum_\tau \epsilon(\tau)c_{\tau_1\tau_2}^{k_1}\cdots c_{\tau_{p-1}\tau_p}^{k_{p/2}}\right),$$

or

$$\sum_{k_1,\cdots,k_{p/2}=r-n+1}^{r}\left(\sum_\sigma \epsilon(\sigma)c_{\sigma_1 c_{\sigma_2}}^{k_1}\cdots c_{\sigma_{p-1}\sigma_p}^{k_{p/2}}\right)^2.$$

This proves the proposition.

Observe that for the usual sectional curvature γ_2 one obtains (directly by the lemma) the expression

$$\gamma_2(x_0;P) = \langle [X_1, X_2], [X_1, X_2]\rangle$$

i.e. $\gamma_2(x_0;P)=0$ if and only if $[X_1, X_2]=0$ (see [6]). Thus by (4) we clearly have the implication $\gamma_2=0\Rightarrow\gamma_p=0$; for all even p with $0<p\leqq n$. This is true for any Riemannian manifold [7, Theorem 6.4].

We remark that our theorem applies in particular to compact Riemannian symmetric spaces equipped with the metric arising naturally from a bi-invariant metric on the group of isometries.

The manifold of a compact Lie group G is with respect to a bi-invariant metric g a Riemannian symmetric space and one can obtain the sectional curvatures by applying the proposition. But in this case it is simpler to observe that the (0)-connection of G [5, p. 49] is the Riemannian connection of g. The expression $R(X, Y)V= -[[X, Y], V]/4$ for the curvature tensor R in e; $X, Y, V\in\mathbf{G}$ [5, p. 49] shows that

$$\langle R(X, Y)V, W\rangle = -\langle [X, Y], [V, W]\rangle/4.$$

By a similar computation as in the proof of the proposition, one obtains for the pth sectional curvature γ_p on a p-plane $P\subset\mathbf{G}$ the expression

$$\gamma_p(e;P) = \frac{1}{2^{3p/2}\cdot p!} \cdot \sum_{k_1,\cdots,k_{p/2}}\left(\sum_\sigma \epsilon(\sigma)\langle[X_{\sigma_1}, X_{\sigma_2}], X_{k_1}\rangle\cdots\langle[X_{\sigma_{p-1}}, X_{\sigma_p}], X_{k_{p/2}}\rangle\right)^2 \tag{7}$$

where $X_1, \cdots, X_n$ is an orthonormal base of $\mathbf{G}$, σ runs through the permutations of $\{1, \cdots, p\}$ and $(\mathrm{k}_1, \cdots, k_{p/2})$ runs now through all $p/2$-tuples of integers k_i with $1\leqq k_i\leqq n$ for $i=1, \cdots, p/2$.

For an even-dimensional group it is clear that $\gamma_n=0$ by the generalized Gauss-Bonnet theorem. (7) gives therefore the identity

$$\sum_{\sigma} \epsilon(\sigma)\langle[X_{\sigma_1}, X_{\sigma_2}], X_{k_1}\rangle \cdots \langle[X_{\sigma_{n-1}}, X_{\sigma_n}], X_{k_{n/2}}\rangle = 0$$

valid for any orthonormal base $X_1, \cdots, X_n$ of $\mathbf{G}$ and any $n/2$-tuple of integers k_i with $1 \leqq k_i \leqq n$ for $i=1, \cdots, n/2$.

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