THE INTERSECTION OF THE FREE MAXIMAL IDEALS IN A COMPLETE SPACE

STEWART M. ROBINSON

Introduction. In [1, p. 123] Gillman and Jerison prove that if X is realcompact, the intersection of the free maximal ideals in $C(X)$ is $C_K(X)$, the ring of functions with compact support. Other authors [2], [3] have proved this result for discrete spaces and p-spaces, respectively. This note provides a proof of the fact that the result holds for any space admitting a complete uniform structure. We point out also that unlike that of the Gillman-Jerison theorem, our proof requires no prior construction of βX, the Stone-Cech compactification of X. Since a realcompact space is complete in the structure generated by $C(X)$, our result extends the Gillman-Jerison theorem. (Whether every complete space is realcompact is an unsettled question: its resolution would require a proof of the existence or nonexistence of measurable cardinals.)

We will employ the same terminology as in [1] and assume all spaces are completely regular and all uniform structures Hausdorff. Also, we take our uniform structures to be defined by pseudometrics. For a topological space X, $C(X)$ denotes its ring of real valued continuous functions, and for $f \in C(X)$, $Z_f = \{x : f(x) = 0\}$. Thus, the support of f is $\mathrm{cl}_X[X - Z_f]$ and $C_K(X)$ is the ring of functions with compact support.

An ideal $I \subseteq C(X)$ is free provided $\bigcap \{Z_f : f \in I\} = \emptyset$ and a maximal ideal which is free is called a free maximal ideal. A subset $S \subseteq X$ is C-embedded in X provided each $f \in C(S)$ admits a continuous extension to all of X.

LEMMA. *If $f \in C(X)$ and if $X - Z_f$ contains a closed, noncompact C-embedded subset, then f fails to belong to some free maximal ideal of $C(X)$.*

PROOF. Let $A \subseteq X - Z_f$ satisfy the hypotheses of the lemma. Then, there exists an $h \in C(X)$ such that $A \subseteq Z_h$ and $h[Z_f] = 1$. Upon noting that h and f cannot belong to the same ideal ($f^2 + h^2$ is a unit), we may complete the proof by showing that h belongs to some free maximal ideal M. Since A is a closed, noncompact subset of Z_h, Z_h is not compact; hence, h belongs to some free ideal I. According

Presented to the Society, December 11, 1964; received by the editors March 1, 1965.

to Zorn's lemma, I is contained in a maximal ideal M and since $\bigcap\{Z_f : f \in M\} \subseteq \bigcap\{Z_g : g \in I\} = \varnothing$, M is free.

THEOREM. *If X admits a complete uniform structure D, then the intersection of the free maximal ideals in $C(X)$ is $C_K(X)$.*

PROOF. $C_K(X)$ is contained within every free ideal, hence in the intersection of all free maximal ideals.

Conversely, suppose $\mathrm{cl}_X[X - Z_f]$ is not compact. As a closed subset of the complete space (X, D), it is complete in the relative structure generated by D; in fact, it is the completion of $X - Z_f$ in the relative structure. Since $X - Z_f$ has a noncompact completion, we infer that for some $d \in D$ and $\rho > 0$, there is an infinite closed set $A \subseteq X - Z_f$ such that for distinct points a and a' in A $d(a, a') \geqq \rho$. For $g \in C(A)$, the function G defined by:

$$G(x) = g(a)(1 - 3d(x, a)/\rho) \qquad \text{if for some } a \in A, \quad d(x, a) \leqq \rho/3$$

$$G(x) = 0 \qquad \text{for other } x$$

is a continuous extension of g to all of X. Thus, A is C-embedded in X and the conclusion follows from the preceding lemma.

This hypothesis that X is complete is not a necessary condition. The space W of all ordinals less than the first uncountable ordinal is pseudocompact, but not compact. Hence it admits no complete uniform structure. Nevertheless, the one free maximal ideal in $C(W)$ is the set $\{f \in C(W) : f$ vanishes on some tail of $W\}$, and this set is precisely $C_K(W)$. (This example is constructed in [1] to show that realcompactness is not a necessary condition.)

REFERENCES

1. L. Gillman and M. Jerison, *Rings of continuous functions*, Princeton Univ. Press, Princeton, N. J., 1960.

2. I. Kaplansky, *Topological rings*, Amer. J. Math. **69** (1947), 153–183.

3. C. W. Kohls, *Ideals in rings of continuous functions*, Fund. Math. **45** (1957), 28–50.

UNION COLLEGE, SCHENECTADY