SOME CHARACTERIZATIONS OF FUNCTIONS OF BAIRE CLASS 1

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Introduction. A function f defined on a topological space X with values in I (the unit interval) is said to be of Baire class 1 provided $f^{-1}(F)$ is a G_δ in X whenever F is closed in I. One easily sees that f is of Baire class 1 if it is the pointwise limit of a sequence of continuous functions from X into I, and, if X is metric, then the converse is true [**3**; p. 280 et seq.].

In this paper we find necessary and sufficient conditions on a graph G of a function $f: I \rightarrow I$ in order that f be of Baire class 1. If G is connected, we obtain the following purely topological condition: (*) G is the intersection of a sequence of simply connected open sets. Simple connectedness cannot be deleted here as is shown by the example in [**2**]. In the case where the graph is not assumed connected, condition (*) is not sufficient (see the example of §3) and an additional nontopological restriction must be placed on the open sets. In the last section we indicate how some of our results can be extended to cover functions with more general domains.

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Preliminaries. We use standard notation for subintervals of I and for points of I^2 (the unit square); whether (x, y) denotes an open subinterval of I or point of I^2 will be clear from the context. The Cartesian product notation is used where convenient; for example,

$$[a, b] \times \{c\} = \{(x, y) \in I^2 \mid a \leqq x \leqq b \text{ and } y = c\}.$$

For each x in I, l_x denotes the vertical interval $\{x\} \times I$.

We say that a subset U of I^2 is simply connected provided U is connected and, if S is a simple closed curve lying in U, then U contains one of the two components of the complement of S in the plane. The following characterization of simple connectedness in I^2 follows easily from Theorem 14, p. 171, of [**4**]: An open subset U of I^2 is simply connected if and only if U is connected and every component of $I^2 - U$ meets the boundary of I^2.

Unless otherwise stated, all functions will be understood to have domain I and range contained in I.

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1. A necessary condition.

DEFINITION. An open subset U of I^2 has *property C* provided that for each x in I, $U \cap l_x$ is of the form $\{x\} \times (a, b)$. Note that a connected subset of I^2 having property C is simply connected.

THEOREM 1. *If f is of Baire class 1 then there is a sequence of open subsets of I^2, each having property C, whose intersection is the graph G of f.*

PROOF. Let n and i be integers such that $0 \leqq i < n$. Select a nested sequence, $\{U(n, i, j) \mid j = 1, 2, \cdots\}$, of open subsets of I whose intersection is $f^{-1}[i/n, (i+1)/n]$. For fixed n and j, define:

$$\begin{aligned} U(n, j) = {} & [U(n, 0, j) \times [0, 2/n)] \cup \cdots \\ & \cup [U(n, k, j) \times ((k-1)/n, (k+1)/n)] \cup \cdots \\ & \cup [U(n, n-1, j) \times ((n-2)/n, 1)]. \end{aligned}$$

Clearly $U(n, j)$ is open in I^2 and contains G. If (x, y) is a point of $\bigcap_{j=1}^{\infty} U(n, j)$ then for some integer k $(0 \leqq k < n)$ and infinitely many j, we have:

$$(x, y) \in U(n, k, j) \times ((k-1)/n, (k+1)/n)$$

(with appropriate modification for $k = 0$ or $n-1$). Since the $U(n, k, j)$ are nested, this implies that $f(x)$ is in $[k/n, (k+1)/n]$; thus $|y - f(x)| < 2/n$. The last inequality implies that $G = \bigcap_{n=1}^{\infty} \bigcap_{j=1}^{\infty} U(n, j)$.

To complete the proof of the theorem it suffices to show that for each pair of positive integers n and j there is a sequence of open sets, each of which contains G and has property C, such that $U(n, j)$ contains the intersection of this sequence.

Fix n and j and suppose i is an integer with $0 \leqq i \leqq n$. Let $A(i) = \{x \in I \mid f(x) > i/n\} \cap \{x \in I \mid (x, i/n) \in I^2 - U(n, j)\}$ and $B(i) = \{x \in I \mid f(x) < i/n\} \cap \{x \in I \mid (x, i/n) \in I^2 - U(n, j)\}$; then A_i and B_i are F_σ sets in I. Let $\{A(i, k) \mid j = 1, 2, \cdots\}$ and $\{B(i, k) \mid k = 1, 2, \cdots\}$ be closed subsets of I such that $\bigcup_{k=1}^{\infty} A(i, k) = A(i)$ and $\bigcup_{k=1}^{\infty} B(i, k) = B(i)$.

For $0 \leqq i \leqq n$ and k arbitrary the sets $\tilde{A}(i, k) = A(i, k) \times [0, i/n]$ and $\tilde{B}(i, k) = B(i, k) \times [i/n, 1]$ are closed in I^2 and miss G. Let $V_k = I^2 - (\bigcup_{i=1}^{n} [\tilde{A}(i, k) \cup \tilde{B}(i, k)])$; then V_k is open in I^2, contains G, and has property C.

It remains to show that $\bigcap_{k=1}^{\infty} V_k \subset U(n, j)$. Suppose (x, y) is in $I^2 - U(n, j)$; by the construction of $U(n, j)$, there exist integers s and t such that $0 \leqq s \leqq t \leqq n$ and $(x, y) \in \{x\} \times [s/n, t/n] \subset I^2 - U(n, j)$. If $f(x) > y$ then $f(x) > t/n$ and there is an integer k such that (x, y)

$\in \tilde{B}(t, k)$; dually, if $f(x) < y$ then, for some k, $(x, y) \in \tilde{A}(s, k)$. In either case we have $(x, y) \in I^2 - V_k$, for some k, and the proof is complete.

2. **The connected case.** We assume throughout this section that f is a function whose graph, G, is connected.

LEMMA. *If U is an open, simply connected subset of I^2 containing G and F is a finite subset of I, then there is a continuous function g such that $g(x) = f(x)$ for each x in F and the graph of g lies in U.*

PROOF. Since G is connected it suffices to prove that there is $y > 0$ such that for $0 \leqq x \leqq y$ there is an arc A_x with endpoints $(0, f(0))$, $(x, f(x))$ such that $A_x \subset U$ and if $0 \leqq x' \leqq x$ then $A_x \cap l_{x'}$ is a single point.

Since U is open, there is $\epsilon > 0$ such that the intersection, V, of the ϵ neighborhood of $(0, f(0))$ with I^2 lies in U. Since G is connected, there is $y > 0$ such that $(y, f(y)) \in V$.

If $0 \leqq x \leqq y$ and $(x, f(x)) \in V$, we let A_x be the straight line segment from $(0, f(0))$ to $(x, f(x))$. If $(x, f(x)) \notin V$, then $0 < x < y$ and (without loss of generality) we may assume $f(x) < f(0)$. We assert that $K = \{x\} \times [f(x), f(0)]$ is contained in U. If not, then some component C of $I^2 - U$ meets K. Since U is simply connected, C also meets the boundary of I^2. Let (x, z) be a point of $K \cap C$. Then $C \cup (\{x\} \times [z, 1])$ misses G and, relative to I^2, separates $(0, f(0))$ from one of the points $(x, f(x))$, $(y, f(y))$. Since this contradicts connectedness of G the assertion is proved. Since K lies in U there is w such that $0 \leqq w < x$ and the straight line segment S from $(w, f(0))$ to $(x, f(x))$ lies in U. We then let $A_x = ([0, w] \times \{f(0)\}) \cup S$.

THEOREM 2. *In order that f be of Baire class 1 it is necessary and sufficient that G be the intersection of a sequence of simply connected open subsets of I^2.*

PROOF. The necessity of the condition follows from Theorem 1. Conversely, suppose $G = \bigcap_{i=1}^{\infty} U_i$ where the U_i are open in I^2 and simply connected. For each positive integer n let V_n denote the component of $U_1 \cap \cdots \cap U_n$ containing the connected set G. It is easy to verify that each V_n is simply connected. Let $\{(x_i, f(x_i)) \mid i = 1, 2, \cdots\}$ be a dense subset of G. For each positive integer n there exists, according to the lemma, a continuous function f_n such that $f_n(x_i) = f(x_i)$, $i = 1, \cdots, n$, and the graph of f_n lies in V_n.

Suppose that for some x in I, the sequence $\{f_n(x)\}$ does not converge to $f(x)$; we may then assume that for some $\epsilon > 0$ and infinitely many j, $f_j(x) > f(x) + \epsilon$. Assume $0 < x < 1$ and denote by z the point

$(x, f(x)+\epsilon)$. Since $\bigcap_{n=1}^{\infty} V_n = G$, and since the V_n are nested, there is an integer N such that for $n \geqq N$, z is in $I^2 - V_n$. Pick a disk D such that $(x, f(x)) \in D \subset V_N$. Pick n, $m \geqq N$ such that $(x_n, f(x_n)) \in D \cap ([0, x) \times I)$ and $(x_m, f(x_m)) \in D \cap ((x, 1] \times I)$. Choose $J \geqq \max(n, m, N)$ such that $f_J(x) > f(x) + \epsilon$.

The component C_J of $I^2 - U_J$ which contains z misses the arc $\{(y, f_J(y)) \mid x_n \leqq y \leqq x_m\}$ but meets the boundary of I^2, hence C_J meets D and, since $C_J \subset I^2 - U_1$, we have: $(I^2 - U_1) \cap D \neq \emptyset$. But this holds for disks D of arbitrarily small diameter, so that $(x, f(x) \in I^2 - U_1$. This contradiction shows that, after all, $\{f_n(x)\}$ converges to $f(x)$ for $0 < x < 1$. We omit the corresponding one sided argument for the case $x = 0$ or 1.

3. **The general case.** We now drop the assumption that the graph of f is connected. Before giving the theorem for the general case we exhibit a function not of Baire class 1 whose graph is the intersection of a nested sequence of simply connected open sets.

Let $\{(r_i, s_i) \mid i = 1, 2, \cdots\}$ be a sequence of pairwise disjoint open intervals in I the complement of whose union is the Cantor ternary set, and let E denote the collection of end-points of these intervals. For each positive integer i, let $f(r_i) = f(s_i) = 1/2 - 1/2^{i+1}$ and, for $x \notin E$, let $f(x) = 1$. Now f is not of Baire class 1 because $f^{-1}[0, 1/2] = E$ and E is not a G_δ set in I^2.

Let n be a positive integer. The collection $\{(x, f(x)) \mid x \in E\}$ is discrete in I^2; hence, for each $x \in E$ there exists an open disk $U(x)$ with center $(x, f(x))$ and radius $\delta(x)$ such that, for each j, $\delta(r_j) = \delta(s_j) \leqq 1/2^n$, and if x and y are distinct points of E, then $U(x) \cap U(y) = \emptyset$. For each j, let t_j denote the midpoint of the interval (r_j, s_j); let W_j denote the union of $U(r_j)$, $U(s_j)$ and the following four open rectangles: $[(r_j, t_j) \cup (t_j, s_j)] \times (f(r_j), f(r_j) + \delta(r_j))$, $[(t_j - \delta(r_j), t_j) \cup (t_j, t_j + \delta(r_j))] \times (f(r_j), 1 - 1/2^{n+1})$. Next, define $V_n = (I \times (1 - 1/2^{n+1}, 1]) - (\{(r_i, 1) \mid i = 1, \cdots, n\} \cup \{(s_i, 1) \mid i = 1, \cdots, n\})$ and, finally, let $U_n = V_n \cup \bigcup_{j=1}^{\infty} W_j$. It is easy to see that U_n is open in I^2, simply connected, contains the graph G of f, and that $\bigcap_{n=1}^{\infty} U_n = G$.

THEOREM 3. *In order that f be of Baire class 1, it is necessary and sufficient that there exist a sequence of open subsets of I^2, each having property C, whose intersection is the graph G of f.*

PROOF. The condition is necessary by Theorem 1.

Conversely, suppose $G = \bigcap_{i=1}^{\infty} U_n$ where each U_n is open and has property C. We may assume that the U_n are nested.

For each x in I, $U_n \cap l_x$ is a nondegenerate open interval in l_x;

write $U_n \cap l_x = \{x\} \times (t_n(x), u_n(x))$. The functions u_n and t_n defined in this way are, respectively, lower and upper semicontinuous. By Theorem 4 of [1], there is a continuous function f_n such that, for for each x, $t_n(x) < f_n(x) < u_n(x)$.

Suppose there is x in I and $\epsilon > 0$ such that for infinitely many j, $f_j(x) > f(x) + \epsilon$. Then for infinitely many j, $\{x\} \times (f(x), f(x)+\epsilon) \subset U_j$. Since the U_j are nested, the inclusion holds for all j and $\bigcap_{j=1}^{\infty} U_j$ contains $\{x\} \times (f(x), f(x)+\epsilon)$ which is absurd. This shows that the f_n converge to f pointwise and completes the proof.

4. **Generalizations.** It is possible to extend Theorems 1 and 3 to the case of more general domains for the function involved. We shall merely state these generalizations; their proofs, except for notation changes, are identical with the ones we have given.

We begin by generalizing property C. Given a topological space X, we say that an open subset U of $X \times I$ has *property C-X* in case $U \cap l_x$ is of the form $\{x\} \times (a, b)$ for each x in X. (Thus "property C-I" is the "property C" used above.)

For convenience in what follows, we shall say that a subset G of $X \times I$ has *property B-X* if and only if G is the intersection of a sequence of open subsets of $X \times I$, each of which has property C-X.

Then the following generalization of Theorem 1 holds:

THEOREM 4. *If X is any topological space and $f: X \to I$ is of Baire class* 1, *then the graph of f has property B-X.*

Generalizing the proof of sufficiency in Theorem 3, we have:

THEOREM 5. *If X is normal and countably paracompact and f is a function on X into I whose graph has property B-X then there is a sequence of continuous functions on X into I converging pointwise to f on X.*

If X is metric, so that one has the characterization of functions of Baire class 1 mentioned in the introduction, then the following generalization of Theorem 3 holds:

THEOREM 6. *A function on X into I is of Baire class* 1 *if and only if its graph has property B-X.*

Theorem 2 appears to be a theorem about plane topology. We have no generalization of it in which the condition on the open sets is purely topological; such a generalization is probably accessible via techniques of algebraic topology.

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