SMALL EIGENVALUES OF LARGE HANKEL MATRICES

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In this note we shall determine the asymptotic behavior as $N \to \infty$ of the smallest eigenvalue of the Hankel matrix

$$H_N = (c_{m+n}) \qquad m, n = 0, \cdots, N.$$

It is assumed that the c_n are the moments of a distribution function $\alpha(x)$ on the finite interval $[a, b]$,

$$c_n = \int_a^b x^n \, d\alpha(x),$$

where $w(x) = \alpha'(x)$ satisfies

$$\int_a^b \frac{\log w(x)}{(x-a)^{1/2}(b-x)^{1/2}} \, dx > -\infty.$$

We shall see that for the smallest eigenvalue λ_N of H_N there is an asymptotic formula of the form

$$\lambda_N \sim \rho N^{1/2} \sigma^{-2N}$$

where ρ and σ are constants which will be explicitly determined. In the case of the Hilbert matrix ($c_m = 1/(m+1)$) a partial result was obtained by Todd in [**3**]. (In certain exceptional cases the exponent $\frac{1}{2}$ must be replaced by $\frac{1}{4}$.) It will be found that σ depends only on the interval $[a, b]$.

It will be assumed throughout that $a+b \geqq 0$. This entails no loss of generality since the Hankel matrix corresponding to the distribution function $-\alpha(-x)$ on $[-b, -a]$ has exactly the same eigenvalues as H_N.

LEMMA 1. *Let $P_n(x)$ $(n = 0, 1, \cdots)$ denote the orthogonal polynomials associated with $\alpha(x)$. Then H_N^{-1} is similar to the matrix whose m, n entry is*

$$a_{m,n} = \frac{1}{2\pi} \int_0^{2\pi} P_m(e^{i\theta}) \, P_n(e^{i\theta})^* \, d\theta, \qquad m, n = 0, \cdots, N.$$

PROOF. Write $P_n(x) = \sum_{i=0}^{n} b_{n,i} x^i$. Then

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$$\delta_{m,n} = \int_a^b P_m(x)P_n(x)\,d\alpha(x) = \sum_{i,j=0}^{N} b_{m,i}c_{i+j}b_{n,j}$$

and so if K_N denotes the matrix

$$\begin{bmatrix} b_{0,0} & 0 & 0 & \cdots & 0 \\ b_{1,0} & b_{1,1} & 0 & \cdots & 0 \\ b_{2,0} & b_{2,1} & b_{2,2} & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ b_{N,0} & b_{N,1} & b_{N,2} & \cdots & b_{N,N} \end{bmatrix}$$

we have $I = K_N H_N K_N^T$. Thus $H_N^{-1} = K_N^T(K_N K_N^T)(K_N^T)^{-1}$. But the m, n entry of $K_N K_N^T$ is

$$\sum_{i=0}^{N} b_{m,i}b_{n,i} = \frac{1}{2\pi}\int_0^{2\pi} P_m(e^{i\theta})\,P_n(e^{i\theta})^*\,d\theta,$$

which proves the lemma.

We shall be concerned now with the asymptotic behavior of $a_{m,n}$ as m, $n \to \infty$. This will turn out to be simple enough to enable us to deduce the asymptotic behavior of the largest eigenvalue of $(a_{m,n})$.

LEMMA 2. *We have, uniformly for z bounded away from the interval $[a, b]$,*

$$P_n(z) \sim (b-a)^{-1/2}\pi^{-1/2}\zeta^n A(\zeta),$$

where

$$\zeta = \frac{2}{b-a}z - \frac{b+a}{b-a} + \left[\left(\frac{2}{b-a}z - \frac{b+a}{b-a}\right)^2 - 1\right]^{1/2}$$

(the square root denoting that branch which is positive for large positive z), $A(\zeta)$ is analytic in $|\zeta| > 1$ and

$$\log|A(\rho e^{i\phi})| = -\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left[w\left(\frac{b-a}{2}\cos t + \frac{b+a}{2}\right)|\sin t|\right] \cdot \frac{\rho^2 - 1}{1 - 2\rho\cos(\phi - t) + \rho^2}\,dt.$$

PROOF. If $a = -1$, $b = 1$ this is Theorem 12.1.2 of [2] if $\alpha(x)$ is absolutely continuous and is Theorem 9.3 of [1] for general α. The case of the interval $[a, b]$ may be reduced to this by a linear change of variable since if $q_n(x)$ are the orthogonal polynomials associated with the distribution function

$$\alpha\left(\frac{b-a}{2}x+\frac{b+a}{2}\right)$$

on $[-1, 1]$ then

$$P_n(x) = q_n\left(\frac{2}{b-a}x - \frac{b+a}{b-a}\right).$$

We omit the details.

In view of Lemma 2 we expect that the asymptotic behavior of $a_{m,n}$ depends on the maximum of $|\zeta(z)|$ as z runs over the unit circle. The next lemma will describe this maximum. It is convenient at this point to distinguish three cases:

Case 1. $a > -b/(1+2b)$.
Case 2. $a = -b/(1+2b)$.
Case 3. $a < -b/(1+2b)$.

LEMMA 3. *The maximum value of* $g(\theta) = |\zeta(e^{i\theta})|$ *is given by*

$$\sigma = \begin{cases} \dfrac{b+a+2}{b-a} + \left[\left(\dfrac{b+a+2}{b-a}\right)^2 - 1\right]^{1/2} & \textit{Cases } 1 \textit{ and } 2, \\ \left(\dfrac{1}{|a|b}+1\right)^{1/2} + \left(\dfrac{1}{|a|b}\right)^{1/2} & \textit{Cases } 2 \textit{ and } 3. \end{cases}$$

In Cases 1 *and* 2 *the maximum occurs at* $\theta = \pi$ (*and only there* mod 2π) *and in Case* 3 *at* $\theta = \pm\theta_0$ (*and only there* mod 2π) *where*

$$\cos\theta_0 = \frac{b+a}{2ab}.$$

Moreover in Case 1 *we have* $g''(\pi) \neq 0$, *in Case* 2 *we have* $g''(\pi) = 0$ *but* $g^{iv}(\pi) \neq 0$, *and in Case* 3 *we have* $g''(\theta_0) \neq 0$.

The proof of the lemma is completely elementary and need not be reproduced here.

LEMMA 4. *There is a constant* A, *depending only on the distribution function* $\alpha(x)$, *such that for all* m, n

$$|a_{m,n}| \leqq \begin{cases} A(m+n+1)^{-1/2}\sigma^{m+n} & \textit{Cases } 1 \textit{ and } 3, \\ A(m+n+1)^{-1/4}\sigma^{m+n} & \textit{Case } 2. \end{cases}$$

PROOF. It follows from Lemma 2 that as long as the unit circle does not intersect the interval $[a, b]$ we have

$$|a_{m,n}| \leqq \text{const} \int_0^{2\pi} g(\theta)^{m+n}\, d\theta$$

and the desired conclusions follow readily from Lemma 3 using standard techniques.

To show that the same estimates hold even if the unit circle does intersect $[a, b]$ let us assume that 1 belongs to the interval but -1 does not. (The case in which they both belong to the interval is more complicated in only a trivial way.) We can write, for any $\epsilon > 0$

$$|a_{m,n}| \leqq \frac{1}{2\pi}\int_{-\epsilon}^{\epsilon} |P_m(e^{i\theta})P_n(e^{i\theta})|\, d\theta + \frac{1}{2\pi}\int_{\epsilon}^{2\pi-\epsilon} |P_m(e^{i\theta})P_n(e^{i\theta})|\, d\theta.$$

Since the asymptotic formula of Lemma 2 holds uniformly for $\epsilon \leqq \theta \leqq 2\pi - \epsilon$, the last integral will satisfy the estimate in the statement of the lemma. To estimate the first integral, denote by R_ϵ the rectangle with vertices $e^{\pm i\epsilon}$, $1 \pm i \tan \epsilon$. This rectangle contains the arc of the unit circle given by $|\theta| \leqq \epsilon$. Since the polynomial $P_m(z)P_n(z)$ has only real zeros (Theorem 3.3.1 of [2]) its maximum absolute value on R_ϵ is attained on the horizontal sides of R_ϵ. On these sides we may apply the asymptotic formula of Lemma 2, and so

$$\limsup_{m+n\to\infty} \max_{R_\epsilon} |P_m(z)P_n(z)|^{1/(m+n)} = g(\epsilon + O(\epsilon^2)).$$

Therefore we have as $m+n \to \infty$

$$\int_{-\epsilon}^{\epsilon} |P_m(e^{i\theta})P_n(e^{i\theta})|\, d\theta = O(t^{m+n})$$

for any $t > g(\epsilon + O(\epsilon^2))$. A little computation shows that $g(2\epsilon) > g(\epsilon + O(\epsilon^2))$ if ϵ is small enough. Thus

$$\int_{-\epsilon}^{\epsilon} |P_m(e^{i\theta})P_n(e^{i\theta})|\, d\theta = O(g(2\epsilon)^{m+n}).$$

But $\sigma > g(2\epsilon)$, again for sufficiently small ϵ (recall that $g(\theta)$ does not attain its maximum σ at $\theta = 0$), and so certainly

$$\int_{-\epsilon}^{\epsilon} |P_m(e^{i\theta})P_n(e^{i\theta})|\, d\theta = o((m + n)^{-1/2}\sigma^{m+n}).$$

This completes the proof of the lemma.

The next lemma gives the asymptotic behavior of $a_{m,n}$ as $m, n \to \infty$. First some more notation. We write

$$\gamma = \begin{cases} \dfrac{|A(\zeta(-1))|^2\sigma^{1/2}}{2^{1/2}\pi^{3/2}|g''(\pi)|^{1/2}(b-a)} & \text{Case 1,} \\[2ex] \dfrac{3^{1/4}\Gamma(\frac{1}{4})|A(\zeta(-1))|^2\sigma^{1/4}}{2^{9/4}\pi^2|g^{iv}(\pi)|^{1/4}(b-a)} & \text{Case 2,} \\[2ex] \dfrac{2^{1/2}|A(\zeta(e^{i\theta}))|^2\sigma^{1/2}}{\pi^{3/2}|g''(\theta_0)|^{1/2}(b-a)} & \text{Case 3,} \end{cases}$$

where $|A(\zeta)|$ is given in Lemma 2 and θ_0 in Lemma 3. We shall write, in Case 3,

$$\operatorname{sgn} \zeta(e^{i\theta_0}) = e^{i\phi_0}.$$

(In Cases 1 and 2, $\operatorname{sgn} \zeta(-1) = -1$.)

LEMMA 5. *The following hold as $m, n \to \infty$ with $m-n$ bounded*:

$$a_{m,n} \sim \gamma(-1)^{m-n}(m+n)^{-1/2}\sigma^{m+n} \qquad \textit{Case } 1,$$

$$a_{m,n} \sim \gamma(-1)^{m-n}(m+n)^{-1/4}\sigma^{m+n} \qquad \textit{Case } 2,$$

$$a_{m,n} = \gamma \cos (m-n)\phi_0(m+n)^{-1/2}\sigma^{m+n} + o((m+n)^{-1/2}\sigma^{m+n}) \qquad \textit{Case } 3.$$

PROOF. Suppose the unit circle does not intersect $[a, b]$. (The case in which it does can be handled just as in the proof of Lemma 4.) Then by Lemma 2,

$$a_{m,n} = \frac{1}{2\pi^2(b-a)} \int_0^{2\pi} \{g(\theta)^{m+n}[\operatorname{sgn} \zeta(e^{i\theta})]^{m-n} |A(\zeta(e^{i\theta}))|^2 + o(g(\theta)^{m+n}\} \, d\theta.$$

In Cases 1 and 2 the maximum of $g(\theta)$ occurs at $\theta = \pi$ (and nowhere else) and the result follows from Lemma 3 using standard techniques. In Case 3 the maximum occurs at $\pm\theta_0$. Since

$$\zeta(e^{-i\theta_0}) = (\zeta(e^{i\theta_0}))^*, \qquad |A(\bar{\zeta})| = |A(\zeta)|$$

the conclusion in this case also follows easily from Lemma 3.

THEOREM. *If λ_N is the smallest eigenvalue of H_N, then as $N \to \infty$*,

$$\lambda_N \sim \gamma^{-1}(\sigma^2 - 1)(2N)^{1/2}\sigma^{-2(N+1)} \qquad \textit{Case } 1,$$

$$\lambda_N \sim \gamma^{-1}(\sigma^2 - 1)(2N)^{1/4}\sigma^{-2(N+1)} \qquad \textit{Case } 2,$$

$$\lambda_N \sim 2\gamma^{-1}\left[\frac{1}{\sigma^2 - 1} + \left(\frac{1}{\sigma^4 - 2\sigma^2 \cos 2\phi_0 + 1}\right)^{1/2}\right]^{-1} (2N)^{1/2}\sigma^{-2(N+1)} \qquad \textit{Case } 3.$$

PROOF. We shall consider in detail only Case 3; the others are easier. Let us write

$$(1)\qquad \begin{aligned} b_{m,n} &= \cos (m - n)\phi_0\sigma^{m+n}, \\ c_{m,n} &= a_{m,n} - \gamma(2N)^{-1/2}b_{m,n}. \end{aligned}$$

Fix N_0 and ϵ. It follows from Lemma 5 that if m and n are sufficiently large, but $|m-n| \leqq N_0$, we shall have

$$\left| a_{m,n} - \gamma \cos(m - n)\phi_0(m + n)^{-1/2}\sigma^{m+n} \right| \leqq \epsilon(m + n)^{-1/2}\sigma^{m+n}.$$

Therefore if both m and n exceed $N-N_0$ and N is sufficiently large we shall have

$$(2)\qquad \begin{aligned} |c_{m,n}| &= \left| a_{m,n} - \gamma \cos(m - n)\phi_0(2N)^{-1/2}\sigma^{m+n} \right| \\ &\leqq \epsilon(m + n)^{-1/2}\sigma^{m+n} + \gamma\sigma^{m+n}[(2N - 2N_0)^{1/2} - (2N)^{1/2}] \\ &\leqq \epsilon N^{-1/2}\sigma^{m+n}. \end{aligned}$$

It follows from Lemma 4 that for all m, n

$$(3)\qquad |c_{m,n}| \leqq A_1(m + n + 1)^{-1/2}\sigma^{m+n}$$

where A_1 is a constant depending only on the distribution function $\alpha(x)$. Denote by μ_N the eigenvalue of largest absolute value of the matrix $(c_{m,n})$ $(m, n=0, \cdots, N)$. Then from (2) and (3) we obtain

$$\begin{aligned} \mu_N^2 &\leqq \sum_{m,n=0}^{N} c_{m,n}^2 \leqq \epsilon N \sum_{m,n=N-N_0}^{N} \sigma^{2(m+n)} + 2A_1^2 \sum_{m=0}^{N-N_0} \sum_{n=0}^{N} \frac{\sigma^{2(m+n)}}{m + n + 1} \\ &\leqq \frac{\epsilon^2\sigma^{4(N+1)}}{(\sigma^2 - 1)^2N} + A_2\frac{\sigma^{2(2N-N_0)}}{2N - N_0}, \end{aligned}$$

where A_2 is another constant. If now N_0 is taken sufficiently large in comparison to ϵ, this will imply for sufficiently large N

$$(4)\qquad |\mu_N| \leqq \frac{2\epsilon\sigma^{2(N+1)}}{(\sigma^2 - 1)N^{1/2}}.$$

Now Lemma 1 implies that λ_N^{-1} is the largest eigenvalue of $(a_{m,n})$ $(m, n=0, \cdots, N)$. It follows therefore from (1) and (4) that if ν_N is the largest eigenvalue of $(b_{m,n})$ $(m, n=0, \cdots, N)$, we have

$$(5)\quad \gamma(2N)^{-1/2}\nu_N - \frac{2\epsilon\sigma^{2(N+1)}}{(\sigma^2 - 1)N^{1/2}} \leqq \lambda_N^{-1} \leqq \gamma(2N)^{-1/2}\nu_N + \frac{2\epsilon\sigma^{2(N+1)}}{(\sigma^2 - 1)N^{1/2}}$$

for sufficiently large N. Since the eigenvectors of $(b_{m,n})$ must be linear combinations $\alpha \cos n\phi_0\sigma^n+\beta \sin n\phi_0\sigma^n$ it is easy to see that

ν_N is the largest eigenvalue of

$$\begin{bmatrix} A & B \\ B & C \end{bmatrix} = \begin{bmatrix} \sum_0^N \cos^2 n\phi_0\sigma^{2n} & \sum_0^N \sin n\phi_0 \cos n\phi_0\sigma^{2n} \\ \sum_0^N \sin n\phi_0 \cos n\phi_0\sigma^{2n} & \sum_0^N \sin^2 n\phi_0\sigma^{2n} \end{bmatrix}.$$

We find that as $N \to \infty$

$$A = \frac{1}{2}\left[\frac{1}{\sigma^2 - 1} + \frac{\sigma^2 \cos 2N\phi_0 - \cos 2(N+1)\phi_0}{\sigma^4 - 2\sigma^2 \cos 2\phi_0 + 1}\right]\sigma^{2(N+1)} + O(1),$$

$$C = \frac{1}{2}\left[\frac{1}{\sigma^2 - 1} - \frac{\sigma^2 \cos 2N\phi_0 - \cos 2(N+1)\phi_0}{\sigma^4 - 2\sigma^2 \cos^2 \phi_0 + 1}\right]\sigma^{2(N+1)} + O(1),$$

$$B = \frac{1}{2}\,\frac{\sigma^2 \sin 2N\phi_0 - \sin 2(N+1)\phi_0}{\sigma^4 - 2\sigma^2 \cos^2 \phi_0 + 1}\,\sigma^{2(N+1)} + O(1),$$

and from these there follows easily

$$(6) \quad \nu_N = \frac{1}{2}\left[\frac{1}{\sigma^2 - 1} + \left(\frac{1}{\sigma^4 - 2\sigma^2 \cos 2\phi_0 + 1}\right)^{1/2}\right]\sigma^{2(N+1)} + O(1).$$

The theorem follows from (6) and (5) if we observe that ϵ was arbitrarily small.

We regret to announce that in the case of the Hilbert matrix

$$\left(\frac{1}{m+n+1}\right) \qquad (m, n = 0, 1, \cdots, N)$$

our result takes the form

$$\lambda_N \sim 2^{9/8}\pi^{3/2}(73 - 48(2)^{1/2})^{-1}N^{1/2}(3 + 2(2)^{1/2})^{-2N-3/4} \quad (N \to \infty).$$

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