THE UNIVALENCE OF FUNCTIONS ASYMPTOTIC TO NONCONSTANT LOGARITHMIC MONOMIALS[1]

E. W. CHAMBERLAIN

Introduction. The title refers to analytic functions $s(x)$ which behave like nonconstant logarithmic monomials $M(x)=cx^{m_0}(\log x)^{m_1} \cdots (\log_r x)^{m_r}$ (where c is a complex number $\neq 0$, the m_i are real, and $\log_N$ is the N-fold iterate of the principal determination of log) in the sense that $s(x)/M(x)\to 1$ as $x\to\infty$ in the complex plane.

DEFINITION. An analytic function E is said to $\to 0$ rapidly enough for M if $E\to 0$ and $(M/M')E'\to 0$ as $x\to\infty$.

Theorem 2 states that if $s=M(1+E)$ where $E\to 0$ rapidly enough for M, then s is 1-1 in some neighborhood of infinity.

The neighborhood bases for ∞ with which we shall be concerned are families $\bar{F}(\alpha, \beta)$ whose elements are sector-like regions $V(\alpha, \beta, \xi)$ defined as follows: Let $-\pi \leqq \alpha < \beta \leqq \pi$. Let $\xi(\delta)$ be a real-valued function defined and bounded below on some subinterval $(0, \gamma)$ of $(0, (\beta-\alpha)/2)$. Let $T(\alpha+\delta, \beta-\delta, \xi(\delta)e^{i\mu})$ be the sector $\{z : \alpha+\delta < \arg(z-\xi(\delta)e^{i\mu}) < \beta-\delta\}$, where $\mu=(\alpha+\beta)/2$. $V(\alpha, \beta, \xi)$ is then $\bigcup\{T(\alpha+\delta, \beta-\delta, \xi(\delta)e^{i\mu}) : 0<\delta<\gamma\}$. The family of all such $V(\alpha, \beta, \xi)$ is denoted $\bar{F}(\alpha, \beta)$. Such neighborhood bases are dealt with in the asymptotic theory of ordinary differential equations in the complex domain, as developed in [1] and [2]. (See [1, p. 44].)

REMARK. Taking $s=M(1+E)$ where $M(x)=\log x$ and $E(x) = x^i/\log x$, it is easily seen that $s/M\to 1$ over $\bar{F}(-\pi, +\pi)$ but s is not univalent on any member of $\bar{F}(-\pi, +\pi)$ (since $s'=0$ infinitely far out on the real axis); $E\to 0$, but not rapidly enough for M in this case.

Theorem 2 justifies a large class of changes of independent variables in the study of differential equations. Functions $f(x)$ (in particular, coefficients of differential equations) can be regarded legitimately as functions $F(s)$, over suitable neighborhoods of ∞, on the condition that s be asymptotic to a nonconstant logarithmic monomial M in the sense described. Changes of this type are important in the treatment of linear differential operators with repeated approximate factors (cf. [3] where such a change is effected by the formal substitution $-W(x)dx=ds$).

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Theorem 1 shows that the class of substitutions effected by writing $ds = V(x)dx$ where $V/V_0 \to 1$, V_0 being a logarithmic monomial such that $\int_{x_0}^{\infty} V_0 = \infty$, is precisely the class of substitutions $s = M(1+E)$, where $E \to 0$ rapidly enough for M and where $M \to \infty$ as $x \to \infty$.

The principal device used in this paper is the Sharp Form of the Generalized Mean Value Theorem, given in Lemma 1. It asserts, roughly, that the tangent vector to a *simple* differentiable arc attains parallelism (as opposed to antiparallelism) to the vector from initial to terminal point. Consequently, a simple closed curve with no more than two points at which the curve is not smooth has tangent vectors whose arguments differ by π. The crux of the univalence discussion is that for the functions $s(x)$ and for the domains $V(\alpha, \beta, \xi)$ in question, the images of certain paths joining arbitrary points in $V(\alpha, \beta, \xi)$ are curves on which the argument of the tangent vector is limited to values differing by less than π—partly because of limitations on $s'(x)$, partly because of the geometry of the path. Thus s cannot map such a path onto a closed curve, and hence assumes two different values at the end points.

Notation. μ will always represent one half the sum of the first two arguments of the function T. For complex z_1, z_2, (z_1, z_2) and $[z_1, z_2]$ will represent the open and closed line segments determined by the z_i.

LEMMA 1 (SHARP FORM OF THE GENERALIZED MEAN VALUE THEOREM). *Let C be a simple arc given by a map $z(t) = (x(t), y(t))$ from $[0, 1]$ into the complex plane which is continuous and* 1-1 *on* $[0, 1]$ *and such that $z'(t)$ exists and is never zero for all $t \in (0, 1)$. Then there exists a $t_1 \in (0, 1)$ such that* $\arg(z'(t_1)) = \arg(z(1) - z(0))$.

PROOF. We discuss the case in which $\arg(z(1) - z(0)) = 0$. No generality is lost in assuming that $C \cap [z(0), z(1)] = \{z(0), z(1)\}$. Defining $z(t) = z(1) + (t-1)(z(0) - z(1))$ for $1 < t \leqq 2$, we may further assume that the simple closed curve $\overline{C}$ given by $z = z(t)$, $0 \leqq t \leqq 2$, is positively oriented. Let $t_1 \in (0, 1) \cup (1, 2)$ be such that $y(t_1) \leqq y(t)$ for all $t \in [0, 2]$. Clearly $\arg(z'(t_1)) = 0$ or $= \pi$. We wish to show that $\arg(z'(t_1)) = 0$.

Since $z'(t_1)$ is a nonzero real number, we can write $\overline{C} = \Lambda \cup (\overline{C} - \Lambda)$ where Λ is a small arc through $z(t_1)$ such that $\Lambda \subset \{z : \arg(z - z(t_1)) \in [0, \pi/4] \cup [3\pi/4, \pi]\} \cup \{z(t_1)\}$. For $z(t) \in \Lambda$, either

(a) $$\arg(z(t) - z(t_1)) \in [0, \pi/4] \quad \text{for } t < t_1$$

and

$$\arg(z(t) - z(t_1)) \in [3\pi/4, \pi] \quad \text{for } t > t_1,$$

or else

(b) $$\arg(z(t)) - z(t_1)) \in [3\pi/4, \pi] \quad \text{for } t < t_1,$$

and

$$\arg(z(t) - z(t_1)) \in [0, \pi/4] \quad \text{for } t > t_1.$$

Suppose (a) is the case—i.e., suppose $\arg(z'(t_1)) = \pi$. Let $q = (X, Y)$ be a point such that $X = x(t_1)$, $Y > y(t_1)$, and sufficiently near $z(t_1)$ that $\{z: \arg(z-q) = -\pi/2\} \cap \overline{C} = \{z(t_1)\}$. As $z(t)$ describes $\overline{C} - \Lambda$ from the left end point of Λ to the right end point of Λ, $\arg(z(t)-q)$ varies from a value near π to a value near 0 (for on $\overline{C}-\Lambda$, $\arg(z(t)-q)$ assumes no value congruent to $3\pi/2$). As $z(t)$ describes Λ from right to left, $\arg(z(t)-q)$ varies from its value near 0 to a value congruent to its initial value; in so doing, $\arg(z(t)-q)$ passes through a value congruent to $-\pi/2$ but through no value congruent to $+\pi/2$. Hence the terminal value of $\arg(z(t)-q)$ equals its initial value minus 2π. This contradicts the assumption that $\overline{C}$ is positively oriented. Therefore (b) is the case—i.e., $\arg(z'(t_1)) = 0$; and $z(t_1) \in C$ since $\arg(z'(t))$ $= \pi$ for $1 < t < 2$.

(I am grateful to Julius S. Dwork for useful suggestions incorporated in this proof.)

LEMMA 2. *Let $z(t)$ map the interval $[a, b]$ continuously into the plane in such a way that $a \leqq t' < t'' \leqq b$, then $z(t') = z(t'')$ if and only if $t' = a$ and $t'' = b$; and suppose that for some $c \in (a, b)$, $z'(t)$ exists and is never zero on $(a, c) \cup (c, b)$. Then there exist $t_1 \in (a, c)$ and $t_2 \in (c, b)$ such that $|\arg(z'(t_1)) - \arg(z'(t_2))| = \pi$.*

PROOF. Apply Lemma 1 to the two simple arcs obtained by restricting z to the intervals $[a, c]$ and $[c, b]$.

LEMMA 3. *Let E be analytic in $V \in \overline{F}(\alpha, \beta)$ and let $E \to 0$ over $\overline{F}(\alpha, \beta)$. Let F be analytic in V and such that $F/W \to 1$ over $\overline{F}(\alpha, \beta)$, where $W(x)$ $= cx^{-1}(\log x)^{-1} \cdots (\log_k x)^{-1+\tau}(\log_{k+1} x)^{a_1} \cdots (\log_{k+r} x)^{a_r}$ with $c \neq 0$ and $\tau > 0$. Let $x_0 \in V$. Then $\int_{x_0}^x EF / \int_{x_0}^x F \to 0$ over $\overline{F}(\alpha, \beta)$.*

PROOF. First we establish

ASSERTION A. Let V_1, B be such that $V_1 \in \overline{F}(\alpha, \beta)$ and $|E(z)| < B$ for all $z \in V_1$. Let $x(r) = re^{i\mu}$ where $\mu = (\alpha+\beta)/2$. Let $\delta \in (0, (\beta-\alpha)/2)$. Then for all sufficiently large r there exists a positive number $S(r, \delta)$ such that $|\int_{x(r)}^x EF / \int_{x(r)}^x F| < 4B$ whenever $|x| > S(r, \delta)$ and $x \in T(\alpha+\delta, \beta-\delta, x(r))$.

PROOF OF ASSERTION A. Let R be a positive number so large that $V_1 \supset T(\alpha+\delta, \beta-\delta, x(R))$, F is analytic on the closure of $T(\alpha+\delta,$

$\beta-\delta,\ x(R))$, and $|F(x)|>\frac{1}{2}\tau|x|^{-1}(\log|x|)^{-1}\cdots(\log_k|x|)^{-1+\tau/2}$ for all $x\in T(\alpha+\delta,\ \beta-\delta,\ x(R))$. We shall show that for each $r\geqq R$ there exists an $S(r,\delta)$ with the prescribed property.

Let $r\geqq R$. It is easily seen that there exists a positive number $S_1\geqq x(r)$ such that for each $\phi\in[\alpha+\delta,\ \beta-\delta]$ there exists a constant $c(\phi)$ such that $|\arg(F(x))-c(\phi)|<\frac{1}{2}$ whenever $|x|\geqq S_1$ and $\arg(x-x(r))=\phi$. Then for each x such that $|x|>S_1$ and $\alpha+\delta<\arg(x-x(r))<\beta-\delta$, if we let $\{x_1\}=[x,\ x(r)]\cap\{z\colon |z|=S_1\}$ and integrate along $[x_1,\ x]$, we obtain

$$\begin{aligned}\left|\int_{x_1}^{x}F(t)\,dt\right| &\\ &=\left|\int_{x_1}^{x}|F(t)|\ \exp[ic(\phi)+i\arg(dt)+i(\arg(F(t))-c(\phi))]\,|dt|\right|\\ &\geqq\int_{x_1}^{x}|F(t)|\ \cos(\arg(F(t)-c(\phi)))\,|dt|>(7/8)\int_{x_1}^{x}|F(t)|\,|dt|\\ &>(7/8)((\log_k|x|)^{\tau/2}-(\log_k|x_1|)^{\tau/2}).\end{aligned}$$

Let B_1 be an upper bound for $\{\int_{x(r)}^{x}|F(t)|\,|dt|\colon x\in\{z\colon |z|=S_1\}\cap T(\alpha+\delta,\beta-\delta,\ x(r))$, integrated over $[x(r),\ x]\}$. Let S_2 be a positive number so large that $(\log_k S_2)^{\tau/2}-(\log_k S_1)^{\tau/2}>2B_1$. Then if x is any member of $T(\alpha+\delta,\ \beta-\delta,\ x(r))$ such that $|x|\geqq S_2$, and $x_1=[x(r),\ x]\cap\{z\colon|z|=S\}$, we have

$$\begin{aligned}\left|\int_{x(r)}^{x}EF\right|&\leqq B\left(\int_{x(r)}^{x_1}|F|\,|dt|+\int_{x_1}^{x}|F|\,|dt|\right)\\ &=B\int_{x_1}^{x}|F|\,|dt|\cdot\left(1+\left(\int_{x(r)}^{x_1}|F|\,|dt|\right)\Big/\left(\int_{x_1}^{x}|F|\,|dt|\right)\right)\\ &\leqq(3B/2)\int_{x_1}^{x}|F|\,|dt|.\end{aligned}$$

At the same time,

$$\left|\int_{x(r)}^{x_1}F(t)\,dt\right|\leqq\int_{x(r)}^{x_1}|F|\,|dt|\leqq B_1$$

and

$$\left|\int_{x_1}^{x}F(t)\,dt\right|>(7/8)\int_{x_1}^{x}|F|\,|dt|>(7/8)\cdot 2B_1,$$

so

$$\left|\int_{x(r)}^{x} F(t)\,dt\right| = \left|\int_{x_1}^{x} F(t)\,dt\right| \cdot \left|1 + \left(\int_{x(r)}^{x_1} F(t)\,dt\right) \Big/ \left(\int_{x_1}^{x} F(t)\,dt\right)\right|$$

$$> (7/8)\left(\int_{x_1}^{x} |F|\,|dt|\right)(1 - B_1/((7/8)2B_1)).$$

Hence $\left|\int_{x(r)}^{x} EF/\int_{x(r)}^{x} F\right| < 4B$ whenever $|x| \geqq S(r, \delta)$ if we define $S(r, \delta) = S_2$.

ASSERTION B. Let $x_{00} = r_0 e^{i\mu} \in V$. Then for each $\epsilon > 0$ and each δ in $(0, (\beta-\alpha)/2)$ there exists a sector $T(\alpha+\delta, \beta-\delta, re^{i\mu})$ such that $\left|\int_{x_{00}}^{x} EF/\int_{x_{00}}^{x} F\right| < \epsilon$ for every $x \in T(\alpha+\delta, \beta-\delta, re^{i\mu})$.

PROOF OF ASSERTION B. Let $\epsilon > 0$ and $\delta \in (0, (\beta-\alpha)/2)$. Since there exists an element of $\overline{F}(\alpha, \beta)$ in which $|E| < \epsilon/16$, Assertion A implies the existence of positive numbers r and S such that $\left|\int_{x(r)}^{x} EF/\int_{x(r)}^{x} F\right| < \epsilon/4$ for each $x \in T(\alpha+\delta, \beta-\delta, x(r)) \cap \{z: |z| > S\}$. We have $\left|\int_{x_{00}}^{x} EF/\int_{x_{00}}^{x} F\right| \leqq N_1/N_0 + N_2/N_0$ where $N_0 = \left|\int_{x(r)}^{x} F\right| \left|1 + \left(\int_{x_{00}}^{x(r)} F/\int_{x(r)}^{x} F\right)\right|$, $N_1 = \left|\int_{x_{00}}^{x(r)} EF\right|$, and $N_2 = \left|\int_{x(r)}^{x} EF\right|$. Let $S' \geqq S$ be so large that whenever $x \in T' = T(\alpha+\delta, \beta-\delta, x(r)) \cap \{z: |z| > S'\}$ the following inequalities hold: (i) $\left|1 + \left(\int_{x_{00}}^{x(r)} F/\int_{x(r)}^{x} F\right)\right| > \frac{1}{2}$, (ii) $\left|\int_{x_{00}}^{x(r)} EF/\int_{x(r)}^{x} F\right| < \epsilon/4$. Let $\bar{r}$ be so large that $T(\alpha+\delta, \beta-\delta, \bar{r}e^{i\mu}) \subset T'$. Then whenever $x \in T(\alpha+\delta, \beta-\delta, \bar{r}e^{i\mu})$ we have $\left|\int_{x_{00}}^{x} EF/\int_{x_{00}}^{x} F\right| < 2(\epsilon/4) + 2(\epsilon/4) = \epsilon$.

Assertion B establishes Lemma 3 for x_0 for the special form x_{00}; to extend the result to arbitrary $x_0 \in V$, write $\int_{x_0}^{x} EF/\int_{x_0}^{x} F = \int_{x_0}^{x_{00}} EF/\int_{x_0}^{x} F + \left(\int_{x_{00}}^{x} EF/\int_{x_{00}}^{x_{00}} F\right)/\left(1 + \left(\int_{x_0}^{x} F/\int_{x_{00}}^{x} F\right)\right)$ and use this limited form of Assertion B together with the fact that $\int_{x_0}^{x} F \to \infty$.

THEOREM 1. *Let $M(x) = cx^{m_0}(\log x)^{m_1} \cdots (\log_r x)^{m_r}$ be a nonconstant logarithmic monomial, with $m_k > 0$ and $m_i = 0$ for $i < k$. Let W be the logarithmic monomial such that $M'/W \to 1$. Then* (a) *if $E \to 0$ rapidly enough for M, $M(1+E)$ can be expressed as an indefinite integral: $M(x)(1+E(x)) = M(x_0)(1+E(x_0)) + \int_{x_0}^{x} W(1+E_0)$, where $E_0 \to 0$; and* (b) *if E_0 is any analytic function which $\to 0$, then $\int_{x_0}^{x} W(1+E_0) = M(1+E)$ where $E \to 0$ rapidly enough for M.*

PROOF. (a) We have $M(x)(1+E(x)) = M(x_0)(1+E(x_0)) + \int_{x_0}^{x} M'(1+E+(M/M')E')$; since $E \to 0$ rapidly enough for M, it follows easily that the integrand can be expressed in the form $W(1+E_0)$.

(b) $W(1+E_0) = M'(1+E_1)$ where $E_1 \to 0$. Therefore $\int_{x_0}^{x} W(1+E_0) = M(x)\left[\left(\int_{x_0}^{x} M'/M(x)\right) + \left(\int_{x_0}^{x} E_1 M'/M(x)\right)\right]$ where the first term in the bracket obviously $\to 1$ while the second term $\to 0$ by Lemma 3. Thus $\int_{x_0}^{x} W(1+E_0) = M(x)(1+E(x))$ with $E \to 0$. We have $E' = A + B$ where $A = (MM' - M'\int_{x_0}^{x} M')/M^2$ and $B = (ME_1M' - M'\int_{x_0}^{x} E_1 M')/M^2$. To show that $E\to 0$ rapidly enough for M, we have $(M/M')A=1-(\int_{x_0}^{x}M'/M(x))\to 0$ and $(M/M')B=E_1-(\int_{x_0}^{x}E_1M'/M(x))$, the latter tending to 0 by Lemma 3.

THEOREM 2. *Let $M(x)$ be a nonconstant logarithmic monomial, as above, with m_k the first nonzero member of $(m_0, \cdots, m_r)$. Let E be analytic and let $E\to 0$ rapidly enough for M over $\overline{F}(\alpha, \beta)$. Let $(\alpha', \beta')\subset(\alpha, \beta)$ and $|m_0(\beta'-\alpha')|\leqq 2\pi$. Then $M(1+E)$ is univalent in some member of $\overline{F}(\alpha', \beta')$.*

PROOF. We shall restrict our attention to the case where $m_k>0$, for it is readily seen that if $m_k<0$ then $M(1+E)=(M_1(1+E_1))^{-1}$ where the first nonzero exponent in M_1 is positive and E_1 is analytic and $E_1\to 0$ rapidly enough for M_1 over $\overline{F}(\alpha, \beta)$.

Expressing $M(1+E)$ as an indefinite integral as in Theorem 1, we shall prove that $s(x)=\int_{x_0}^{x}W(1+E_0)$ is univalent in some member of $\overline{F}(\alpha', \beta')$. The following cases will be discussed in detail:

Case 1. $\beta-\alpha\leqq\pi$; $k>0$, or $k=0$ and $0<m_k\leqq 1$;

Case 2. $\beta-\alpha>\pi$; $k>0$, or $k=0$ and $0<m_k\leqq 1$. The remaining case, in which $k=0$ and $m_0>1$, may be treated similarly, but the details are more complicated. We shall omit this complicated treatment and dispose of this case as follows: Write $M(1+E)=[\tilde{M}(1+\tilde{E})]^{m_0}$ where $\tilde{M}(x)=c^{1/m_0}x^1(\log x)^{m_1/m_0}\cdots(\log_r x)^{m_r/m_0}$ and $\tilde{E}=(1+E)^{1/m_0}-1$. Restricting x, from the outset, to a member of $\overline{F}(\alpha, \beta)$ in which $|E|<1$ and defining $(1+E)^{1/m_0}=\exp((1/m_0)\log(1+E))$ (using the principal value of log), we have $\tilde{E}\to 0$ over $\overline{F}(\alpha, \beta)$. $\tilde{E}\to 0$ rapidly enough for $\tilde{M}$, i.e. $x\tilde{E}'\to 0$, automatically in this case, by Lemma 4. By the validity of the present theorem in Cases 1 and 2, $\tilde{M}(1+\tilde{E})$ is univalent in a member of $\overline{F}(\alpha, \beta)$, hence in a member of each $\overline{F}(\alpha', \beta')$. It remains only to show that $\tilde{M}(1+\tilde{E})$ maps a member of each $\overline{F}(\alpha', \beta')$ into a region in which $z\to z^{m_0}$ is univalent, and this is done in Lemma 5.

For Cases 1 and 2 we construct an element $V(\alpha, \beta, \xi)$ of $\overline{F}(\alpha, \beta)$ in which s is univalent by defining the function $\xi(\delta)$ as follows: For a suitable subinterval $(0, \gamma)$ of $(0, (\beta-\alpha)/2)$, and for each $\delta\in(0, \gamma)$, $\xi(\delta)$ is chosen to be a positive number so large that for all $x\in T_\delta=T(\alpha+\delta, \beta-\delta, \xi(\delta)e^{i\mu})$,

$$\left|\arg(x^{1-m_0}W(x)(1+E_0(x)))-\arg(c)\right|<\delta/4.$$

(This is possible because of obvious properties of iterated logarithms and because $E_0\to 0$ over $\overline{F}(\alpha, \beta)$.) Then s is shown to be univalent in $V(\alpha, \beta, \xi)$ by applying Lemma 2 as follows: For each pair x_1, x_2 in $V(\alpha, \beta, \xi)$ we construct a map $x(t)$, $0\leqq t\leqq 2$, by choosing a third point x_3 in a manner depending on circumstances and defining $x(t) = x_1+t(x_3-x_1)$ for $0\leqq t\leqq 1$ and $x(t)=x_3+(t-1)(x_2-x_3)$ for $1<t\leqq 2$. We define $z(t)=s(x(t))$, and $F(t_1, t_2)=\arg(z'(t_1))-\arg(z'(t_2))$ for $(t_1, t_2)\in J\times J$ where $J=(0, 1)\cup(1, 2)$. Then we show that $|F|<\pi$ on $J\times J$. Lemma 2 implies that $z(t)$ maps no subinterval of $[0, 2]$ onto a simple closed curve, whence it follows that $s(x_1)\neq s(x_2)$.

Since $|F(t_1, t_2)|$ is symmetric in t_1 and t_2, we shall only consider pairs $(t_1, t_2)\in J\times J$ such that $t_1<t_2$.

Notation. (1) Let $\nu=m_0-1$.

(2) Let $F_1=F_1(t_1, t_2)=\nu(\arg(x(t_1))-\arg(x(t_2)))$.

(3) Let $F_2=F_2(t_1, t_2)=\arg(x'(t_1))-\arg(x'(t_2))$.

(4) Let $F_3=F_3(t_1, t_2)=A_1-A_2$, where $A_i=\arg[(x(t_i))^{-\nu}W(x(t_i))(1+E_0(x(t_i)))]-\arg(c)$, so that $F=F_1+F_2+F_3$.

Case 1. Consider $\xi(\delta)$ to be defined for $0<\delta<(\beta-\alpha)/2$. Let x_1 and $x_2\in V(\alpha, \beta, \xi)$; let δ_1, δ_2 be such that $x_i\in T_{\delta_i}$, $i=1, 2$.

(1a) Suppose $[x_1, x_2]\subset T_{\delta_1}\cup T_{\delta_2}$. Let $x_3\in(x_1, x_2)$. Then if t_1 and t_2 are such that $\{x(t_1), x(t_2)\}\subset T_{\delta_j}$ for $j=1$ or $j=2$, we have $|F_1|<\beta-\alpha-2\delta_j$, $F_2=0$, and $|F_3|<\delta_j/2$, so $|F|<\pi$. For any other $(t_1, t_2)\in J\times J$ we have $|F_1|<\beta-\alpha-\delta_1-\delta_2$, $F_2=0$, and $|F_3|<\delta_1/4+\delta_2/4$; thus $|F|<\pi$ on $J\times J$.

(1b) In the contrary subcase we may suppose $x_1\in T_{\delta_1}-T_{\delta_2}$ while $x_2\in T_{\delta_2}-T_{\delta_1}$, $\delta_1>\delta_2$, and $\arg(x_1)>\arg(x_2)$. (The other possibilities lead to similar discussions.) In this situation there exists a point $x_3\in T_{\delta_1}\cap T_{\delta_2}$ such that $\arg(x_3-x_1)=\alpha+\delta_1$ and $\arg(x_2-x_3)=\alpha+\delta_2$. From (1a) we see that $|F|<\pi$ on $(0, 1)\times(0, 1)\cup(1, 2)\times(1, 2)$. For $t_1<1<t_2$ we have $0\geqq F_1>(\alpha+\delta_2)-(\beta-\delta_1)$, $F_2=\delta_1-\delta_2$, and $|F_3|<\delta_1/4+\delta_2/4<\delta_1/2$, so $\delta_1>\delta_1-\delta_2\geqq F_1+F_2>-(\beta-\alpha)+2\delta_1$, whence $\pi>3\delta_1/2>F>-(\beta-\alpha)+3\delta_1/2>-\pi$, so $|F|<\pi$ on $J\times J$.

Case 2. Consider $\xi(\delta)$ to be defined on $(0, \gamma)\subset(0, (\beta-\alpha-\pi)/2)$. Let $x_i\in T_{\delta_i}$, $i=1, 2$, where $\{\delta_1, \delta_2\}\subset(0, \gamma)$.

(2a) Suppose $\arg(x_1)=\arg(x_2)$. Take $x_3\in(x_1, x_2)$. Then $F_1=F_2=0$ and $|F_3|<\max(\delta_1/2, \delta_2/2)<\pi$, so $|F|<\pi$ on $J\times J$.

(2b) Next suppose x_1 and x_2 are such that $[x_1, x_2]\subset T_\delta$ for some $\delta\in(0, \gamma)$ and that $0<\arg(x_1)-\arg(x_2)<\pi-\delta$. Take $x_3\in(x_1, x_2)$. Then $0\geqq F_1>-\pi+\delta$, $F_2=0$, $|F_3|<\delta/2$, and we have $|F|<\pi$ on $J\times J$.

(2c) Next suppose x_1 and $x_2\in T_\delta$ and $\arg(x_1)>\arg(x_2)$, and that $\arg(x_1)-\arg(x_2)\geqq\pi-\delta$ or $[x_1, x_2]\not\subset T_\delta$. Then x_1 and x_2 lie on opposite sides of the line $\{re^{i\mu}: -\infty<r<+\infty\}$. Let $x_3(r)=re^{i\mu}$ and let $b(r) = \arg(x_3(r)-x_1)-\arg(x_2-x_3(r))$. For $r=\xi(\delta)$, $b(r)<(\beta-\delta-\pi)-(\alpha+\delta)<\pi-2\delta$, while $b(r)\to\pi$ as $r\to+\infty$. Hence there exists an $\bar{r}>\xi(\delta)$ such that $b(\bar{r})=\pi-\delta$. Let $x_3=x_3(\bar{r})$. Observing that $\arg(x_1)-\arg(x_3)<\pi-\delta$ and $\arg(x_3)-\arg(x_2)<\pi-\delta$, we see from (2b) that

$|F| < \pi$ on $(0, 1)\times(0, 1)\cup(1, 2)\times(1, 2)$. Now consider the situation in which $t_1 < 1 < t_2$. Here $0 \geqq F_1 > -(\beta-\alpha-2\delta) > -2\pi+2\delta$, $F_2 = \pi-\delta$, and $|F_3| < \delta/2$. Hence $|F| < \pi$ on $J\times J$.

(2d) Finally suppose $x_1 \in T_{\delta_1} - T_{\delta_2}$, $x_2 \in T_{\delta_2} - T_{\delta_1}$, where $\arg(x_1) > \arg(x_2)$, and assume $\delta_1 > \delta_2$ with $\xi(\delta_1) < \xi(\delta_2)$ to fix ideas. Let $\{\bar{x}_3\} = \{z: \arg(z-x_1) = \beta-\delta_1-\pi\} \cap \{z: \arg(z-x_2) = \alpha+\delta_2-\pi\}$. Let $x_3(r) = r\bar{x}_3$ for $r \geqq 1$. Then $[x_i, x_3(r)] \subset T_{\delta_i}$ whenever $r \geqq 1$ $(i = 1, 2)$. Let $b(r) = \arg(x_3(r)-x_1) - \arg(x_2-x_3(r))$. $b(1) = \beta-\alpha-\delta_1-\delta_2-\pi \leqq \pi-\delta_1-\delta_2$, while $b(r) \to \pi$ as $r \to +\infty$. Hence there exists an $\bar{r} > 1$ such that $b(\bar{r}) = \pi-(\delta_1+\delta_2)/2$. Let $x_3 = x_3(\bar{r})$. From (2a)–(2c) it follows that $|F| < \pi$ on $(0, 1)\times(0, 1)\cup(1, 2)\times(1, 2)$. For $t_1 < 1 < t_2$ we have $0 \geqq F_1 > -(\beta-\alpha-\delta_1-\delta_2)$, $F_2 = \pi-(\delta_1+\delta_2)/2$, and $|F_3| < \delta_1/4+\delta_2/4$, whence $|F| < \pi$ on $J\times J$.

Thus s is univalent on $V(\alpha, \beta, \xi)$.

Lemma 4. *If E is analytic and $E \to 0$ over $\overline{F}(\alpha, \beta)$, then $xE'(x) \to 0$ over $\overline{F}(\alpha, \beta)$.*

Proof. Let $\epsilon > 0$. For each $\delta \in (0, (\beta-\alpha)/4)$ let V_δ be an element of $\overline{F}(\alpha, \beta)$ such that $|E(x)| < (\epsilon \sin(\delta))/2$ on V_δ. Let $T(\alpha+\delta, \beta-\delta, x_0) \subset V_\delta$, where $x_0 = r_0 e^{i\mu}$. Let $T = T(\alpha+2\delta, \beta-2\delta, x_0)$. Then for $x \in T$ we have, by the Cauchy integral formula,

$$|xE'(x)| < \tfrac{1}{2}(\epsilon \sin(\delta))(|x|/(|x-x_0| \sin(\delta))).$$

Let $S = \{z: |z/(z-x_0)| < 2\}$. Then $|xE'(x)| < \epsilon$ on $S\cap T$, and it is clear that $S\cap T$ contains a sector $T(\alpha+2\delta, \beta-2\delta, re^{i\mu})$. Let $T(\epsilon, \delta)$ be such a sector. Then $\bigcup\{T(\epsilon, \delta): 0 < \delta < (\beta-\alpha)/4\}$ is an element of $\overline{F}(\alpha, \beta)$ in which $|xE'(x)| < \epsilon$.

Lemma 5. *Let $\tilde{M}(x) = x^1(\log x)^{n_1} \cdots (\log_r x)^{n_r}$ and let $\tilde{E}(x) \to 0$ over $\overline{F}(\alpha, \beta)$. Then $\tilde{M}(1+\tilde{E})$ maps some element of $\overline{F}(\alpha, \beta)$ into the sector $T(\alpha, \beta, 0)$.*

Proof. For each $\delta > 0$ take $r_\delta > 0$ to be so large that for $x \in T_\delta = T(\alpha+\delta, \beta-\delta, r_\delta e^{i\mu})$, $|\arg(x) - \arg(\tilde{M}(x)(1+\tilde{E}(x)))| < \delta$. Then $\tilde{M}(1+\tilde{E})$ maps $\bigcup T_\delta$ into $T(\alpha, \beta, 0)$.

References

1. Walter Strodt, *Contributions to the asymptotic theory of ordinary differential equations in the complex domain*, Mem. Amer. Math. Soc., No. 13 (1954), 81 pp.

2. ———, *Principal solutions of ordinary differential equations in the complex domain*, Mem. Amer. Math. Soc., No. 26 (1957), 107 pp.

3. E. W. Chamberlain, *Differential operators approximately of the form* $(1-(1/W(x))d/dx)^n$, Proc. Amer. Math. Soc. **16** (1965), 725–730.

University of Vermont