THE LACK OF SELF-ADJOINTNESS IN THREE POINT BOUNDARY VALUE PROBLEMS

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In a paper with the same title, to appear shortly in the Pacific Journal of Mathematics, J. W. Neuberger showed that the problem:

$$(py') - qy = g \text{ with } A\begin{bmatrix} (ya) \\ p(a)y'(a) \end{bmatrix} + B\begin{bmatrix} y(b) \\ p(b)y'(b) \end{bmatrix} + C\begin{bmatrix} y(c) \\ p(c)y'(c) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{where} \quad a < b < c,\ p(t) \neq 0$$

for t in $[a, c]$, p, q, g are real continuous functions on $[a, c]$ and A, B, C are real 2×2 matrices; in the determinate case, has a nonsymmetric Green's function if $B\neq 0$. The purpose of this paper is to generalize his result to higher order problems.

Consider problems:

(*) $Y' = FY$ with $A_1Y(a) + B_1Y(b) + C_1Y(c) = 0$ where A_1, B_1, C_1 and $k\times k$ matrices, $k>1$, Y is a k-dimensional vector and $F = (f_{ij})$ satisfies

$$(1)\qquad \begin{array}{l} f_{ij} \in C_{[a,c]},\ f_{ij} = 0 \text{ if } j > i+1 \text{ or } i+j \text{ is even}, \\ f_{i,i+1}(t) \neq 0 \text{ for } t \in [a, c], \qquad i, j = 1, \cdots, k \text{ and} \end{array}$$

(**) $X' = HX$ with $A_2(X)(a) + B_2X(b) + C_2X(c) = 0$ where $H = (f^*_{k+1-j,k+1-i})$, A_2, B_2, C_2 are $k\times k$ matrices. Observe that H also satisfies (1).

In [1] the author has shown that every equation of the form $Ly = \sum_{i=1}^{k} p_i y^{(i)} = 0$ with $p_i \in C_{[a,c]}$, $p_k(t) \neq 0$, $t\in[a, c]$, has a vector-matrix representation of the type $Y' = FY$ with F satisfying (1), and that $X' = HX$ serves as an "adjoint" equation.

Let $M = (m_{ij})$ and $N = (n_{ij})$ denote the unique matrix functions (see [31]) such that $M(t, u) = I + \int_u^t F(s)M(s, u)\, ds$ and $N(t, u) = I + \int_u^t H(s)N(s, u)\, ds$ for all t, u in $[a, c]$. Throughout this paper we will assume that $A = [A_1 + B_1M(b, a) + C_1M(c, a)]^{-1}$ and $B = [A_2 + B_2N(b, a) + C_2N(c, a)]^{-1}$ exist. The existence of these inverses is equivalent to problems (*) and (**), respectively, having no nontrivial solutions. (The proof of this fact is entirely analogous to the 2-point case, see [4], and is therefore omitted.)

Let $V = -AC_1M(c, a)$, $W = -AB_1M(b, a)$, $U = -BC_2N(c, a)$, $Z = -BB_2N(b, a)$.

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Define

$$(2)\quad L(t,x)=\begin{cases}M(t,a)VM(a,x) & b<x\leqq c,\ a\leqq t\leqq c,\\ M(t,a)(V+W)M(a,x) & a\leqq x\leqq b,\ a\leqq t\leqq c;\end{cases}$$

$$(3)\quad J(t,x)=\begin{cases}N(t,a)UN(a,x) & b<x\leqq c,\ a\leqq t\leqq c,\\ N(t,a)(U+Z)N(a,x) & a\leqq x\leqq b,\ a\leqq t\leqq c;\end{cases}$$

$$(4)\quad K(t,x)=\begin{cases}L(t,x) & a\leqq t<x\leqq c,\\ L(t,x)+M(t,x) & a\leqq x\leqq t\leqq c;\end{cases}$$

$$(5)\quad G(t,x)=\begin{cases}J(t,x) & a\leqq t<x\leqq c,\\ J(t,x)+N(t,x) & a\leqq x\leqq t\leqq c.\end{cases}$$

We remark that K is a kernel function for (*) i.e. for any continuous vector G the solution $Y=(y_i)$ of $Y'=FY+G$ with $A_1Y(a)+B_1Y(b)+C_1Y(c)=0$ is given by $Y(t)=\int_a^c K(t,s)G(s)\,ds$ for t in $[a,c]$. The proof, again, is analogous to the 2-point case given in [**4**]. Observe that for

$$G=\begin{bmatrix}0\\0\\ \cdot\\ \cdot\\0\\g\end{bmatrix},\qquad y_1(t)=\int_a^c K_{1k}(t,s)g(s)\,ds,$$

so that K_{1k} is the Green's function of an ordinary boundary value problem.

THEOREM 1. *If either $B_1\neq 0$ or $B_2\neq 0$ then there exist t, x in $[a,c]$ such that*

$$K_{1k}(t,x)\neq(-1)^k(\overset{*}{G}_{1k}(x,t)).$$

The following results are used in the proof of Theorem 1.

$$(6)\qquad N(u,t)N(t,x)=N(u,x)\quad\text{for any } t,u,x\in[a,c]$$

(see [3]).

$$(7)\ m_{ij}(t,x)=(-1)^{i+j}\overset{*}{n}_{k+1-j,k+1-i}(x,t)\ \text{ for } x,t \text{ in } [a,c],\ i,j=1,\cdots,k.$$

(See [2], p. 12.)

(8) For a given $u\in[a,c]$, $m_{1i}(t,u)$, $i=1,\cdots,k$ are linearly independent on any subinterval of $[a,c]$.

PROOF OF THEOREM 1. Suppose $K_{1k}(t,x)=(-1)^k\overset{*}{G}_{1k}(x,t)$ for all

t, x in $[a, c]$. Then, using (2), (3), and (7) $L_{1k}(t, x) = (-1)^k[J^*_{1k}(x, t) + n^*_{1k}(x, t)]x$, $t \in [a, c]$. For $a \leqq x \leqq b$, $b \leqq t \leqq c$, using (2), (3), (6), and (7), we have

$$\sum_{i=1}^{k} m_{1i}(t, a) \sum_{j=1}^{k} (v_{ij} + w_{ij}) m_{jk}(a, x)$$

$$= (-1)^k \sum_{i=1}^{k} (-1)^{1+i} m_{1i}(t, a) \sum_{=1}^{k} (-1)^{j+k} m_{j,k}(a, x) \overset{*}{u}_{k+1-j,k+1-i}$$

$$- \sum_{i=1}^{k} m_{1i}(t, a) m_{ik}(a, x).$$

Hence, using 8, $w_{ij} = (-1)^{i+j+1} u^*_{k+1-j,k+1-i} - \delta_{ij} - v_{ij}$. Similarly for $a < x \leqq c$ and $b < t \leqq c$ we get $v_{ij} = (-1)^{i+j+1} u^*_{k+1-j,k+1-i} - \delta_{ij}$. Hence $W = 0$ and $B_1 = 0$.

Similar considerations lead to $B_2 = 0$. This contradiction completes the proof.

Theorem 1 can be generalized as follows: Instead of (*) and (**) consider

I. $Y' = FY$ with $\sum_{i=1}^{m} A_i Y(a_i) = 0$ where A_i, $i = 1, \cdots, m$, $m > 2$, are $k \times k$ matrices and $a = a_1 < a_2 < \cdots < a_m = c$ and

II. $X' = HX$ with $\sum_{i=1}^{m} B_i X(a_i) = 0$; B_i, $i = 1, \cdots, m$ are $k \times k$ matrices. Construct kernel functions K and G, for Problems I and II, respectively, analogously to (4) and (5). We have

THEOREM 2. *If at least one of A_i, B_i, $i = 2, \cdots, m-1$ is not zero, then there exist t, $u \in [a, c]$ such that $K_{1k}(t, u) \neq (-1)^k (G^*_{1k}(u, t))$.*

The proof is similar and is therefore omitted.

In the rest of the paper we develop similar results for the Green's matrices K and G as a whole—rather than just the upper right hand corner K_{1k} and G_{1k}.

Henceforth $F = (fij)$ is any $k \times k$ matrix of continuous complex valued functions. Let $H = ((-1)^{i+j} \bar{f}_{k+1-j,k+1-i})$; M, N, K and G are defined as above. Let $T = ((-1)^i \delta_{i,k+1-j})$.

THEOREM 3. *If either $B_1 \neq 0$ or $B_2 \neq 0$, then there exist t, x in $[a, c]$ such that*

$$K(t, x) \neq -T^{-1} G^*(x, t) T.$$

NOTE. In the proof of Theorem 3 we use the fact that $M(t,x) = T^{-1} N^*(x, t) T$—a restatement of (7). This is proven in [2, p. 12] only under a more restrictive condition on F. However, the extension of the proof to the present setting is straightforward—the pertinent fact being that $H=-T^{-1}F^*T$.

PROOF OF THEOREM 3. Suppose $K(t, x)=-T^{-1}G^*(x, t)T$ for every t, x in $[a, c]$.

For $a<x<b$ and $b<t<c$ we have

$$\begin{aligned} K(t, x) &= L(t, x) + M(t, x) = M(t, a)(V + W)M(a, x) + M(t, x) \\ &= M(t, a)[V + W + I]M(a, x), \\ G(x, t) &= N(x, a)UN(a, t). \end{aligned}$$

A simple computation, see (7) above, yields

$$-T^{-1}G^*(x, t)T = M(t, a)[-T^{-1}U^*T\,]M(a, x).$$

Hence $V+W+I=-T^{-1}U^*T$.

On the other hand for $b<t<c$, $b<x<c$, we have

$$\begin{aligned} K(t, x) &= L(t, x) = M(t, a)VM(a, x), \\ G(x, t) &= J(x, t) + N(x, t) = N(x, a)UN(a, t) + N(x, t). \end{aligned}$$

A similar computation yields:

$$-T^{-1}G^*(x, t)T = M(t, a)[-T^{-1}U^*T-I]M(a, x).$$

Hence $V=-T^{-1}U^*T-I$; and therefore $W=0=B_1$. Similarly one can show that $B_2=0$. This contradiction completes the proof.

Theorem 3 readily extends to n point problems for any positive integer $n>2$.

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